# DYNAMICS OF RATIONAL DIFFERENCE EQUATION 

$$
X_{N+1}=\frac{\alpha+\beta X_{N}+\gamma X_{N-K}}{B X_{N}+C X_{N-K}}
$$

# USING MATHEMATICAL AND COMPUTATIONAL APPROACH 

By<br>Abed Elrazzaq Alawneh

Supervisor<br>Professor Mohammad Saleh

# BIRZEIT UNIVERSITY <br> FACULTY OF GRADUATE STUDIES 

Dynamics of Rational Difference Equation

$$
x_{n+1}=\frac{\alpha+\beta x_{n}+\gamma x_{n-k}}{B x_{n}+C x_{n-k}}
$$

Using Mathematical And Computational Approach

By
Abed Elrazzaq Alawneh

A Thesis Submitted in Partial Fulfillment of the requirement for MasterDegree in Scientific Computing From the Faculty of Graduate Studies at Birzeit University.

Supervisor<br>Professor Mohammad Saleh

Birzeit, Palestine

Augest 2007

# DYNAMICS OF RATIONAL DIFFERENCE EQUATION 

$$
X_{N+1}=\frac{\alpha+\beta X_{N}+\gamma X_{N-K}}{B X_{N}+C X_{N-K}}
$$

## USING MATHEMATICAL AND COMPUTATIONAL APPROACH

By<br>Abed Elrazzaq Alawneh

This thesis was successfully defended on September 13, 2007 and approved by:

Committee Members
Professor Mohammad Saleh
Dr. Wasfi Alkafri
Dr. Hasan Yousef

Signature
$\qquad$
$\qquad$
$\qquad$

To My Parents

## Table of Contents

Table of Contents ..... v
List of Tables ..... viii
List of Figures ..... ix
Acknowledgements ..... i
Introduction ..... 1
1 Preliminary and Basic Theory Of Difference Equation ..... 6
1.1 Introduction ..... 6
1.2 Difference Equations ..... 8
1.3 Solution of Linear first order Difference Equations ..... 10
1.4 Solutions of Difference Equations of Higher Order ..... 12
1.4.1 Solutions of $k^{\text {th }}$ order homogeneous linear difference with con- stant coefficients ..... 13
1.4.2 Solutions of $k^{\text {th }}$ order nonhomogeneous linear difference with constant coefficients ..... 19
1.5 Solution of Nonlinear Difference Equations ..... 21
1.6 Behavior of Solutions of Difference Equations ..... 27
1.6.1 Equilibrium Points of Difference Equations ..... 28
1.6.2 Stability Theory ..... 29
1.6.3 Graphical Iteration ..... 30
1.7 Criteria for Stability ..... 34
2 Preliminary and Basic Theory Of Rational Difference Equations ..... 36
2.1 Rational Difference Equations ..... 36
2.2 Definitions ..... 37
2.3 Theorems ..... 39
3 Dynamics of $x_{n+1}=\frac{\alpha+\beta x_{n}+\gamma x_{n-k}}{B x_{n}+C x_{n-k}}$ ..... 46
3.1 Change of variables ..... 46
3.2 Equilibrium Points ..... 48
3.3 Linearization ..... 49
3.4 Local Stability ..... 51
3.5 Invariant Intervals ..... 56
3.6 Existence of two cycles ..... 58
3.7 Analysis of Semicycle and oscillation ..... 62
3.8 Global Stability Analysis ..... 66
4 The Special Cases $\alpha \beta \gamma B C=0$ ..... 68
4.1 One parameter $=0$ ..... 68
4.1.1 Dynamics of $x_{n+1}=\frac{\beta x_{n}+\gamma x_{n-k}}{B x_{n}+C x_{n-k}}$ ..... 69
4.1.2 Dynamics of $x_{n+1}=\frac{\alpha+\gamma x_{n-k}}{B x_{n}+C x_{n-k}}$ ..... 71
4.1.3 Dynamics of $x_{n+1}=\frac{\alpha+\beta x_{n}}{B x_{n}+C x_{n-k}}$ ..... 72
4.1.4 Dynamics of $x_{n+1}=\frac{\alpha+\beta x_{n}+\gamma x_{n-k}}{C x_{n-k}}$ ..... 73
4.1.5 Dynamics of $x_{n+1}=\frac{\alpha+\beta x_{n}+\gamma x_{n-k}}{B x_{n}}$ ..... 74
4.2 Two parameters are zero ..... 74
4.2.1 Dynamics of $x_{n+1}=\frac{\gamma x_{n-k}}{B x_{n}+C x_{n-k}}$ ..... 75
4.2.2 Dynamics of $x_{n+1}=\frac{\beta x_{n}}{B x_{n}+C x_{n-k}}$ ..... 76
4.2.3 Dynamics of $x_{n+1}=\frac{\beta x_{n}+\gamma x_{n-k}}{C x_{n-k}}$ ..... 77
4.2.4 Dynamics of $x_{n+1}=\frac{\beta x_{n}+\gamma x_{n-k}}{B x_{n}}$ ..... 78
4.2.5 Dynamics of $x_{n+1}=\frac{\alpha}{B x_{n}+C x_{n-k}}$ ..... 78
4.2.6 Dynamics of $x_{n+1}=\frac{\alpha+\gamma x_{n-k}}{B x_{n}}$ ..... 83
4.2.7 Dynamics of $x_{n+1}=\frac{\alpha+\beta x_{n}}{C x_{n-k}}$ ..... 86
4.3 Three parameters are zero ..... 89
5 Computational Approach ..... 92
5.1 Numerical Examples ..... 92
5.2 Phase Space Diagram ..... 101
5.3 Matlab Program ..... 103
A Appendix ..... 105
A. 1 Rational Difference Equation Program ..... 105
A. 2 Phase Space Diagram Program ..... 107
A. 3 Cobweb Diagram Program ..... 109
Bibliography ..... 111

## List of Tables

5.1 Solution of DE $y_{n+1}=\frac{1+y_{n}+2 y_{n-2}}{y_{n}+2 y_{n-2}}$ ..... 93
5.2 Solution of DE $y_{n+1}=\frac{2+y_{n}+3 y_{n-3}}{y_{n}+4 y_{n-3}}$ ..... 95
5.3 Solution of $y_{n+1}=\frac{2}{y_{n}}+\frac{3}{y_{n-2}}$ ..... 97
5.4 Solution of $y_{n+1}=\frac{3+y_{n}+7 y_{n-3}}{y_{n}+0.4 y_{n-3}}$ ..... 98
5.5 Solution of DE $x_{n+1}=\frac{5}{x_{n-2}}$ ..... 100

## List of Figures

1.1 Solution of $x_{n+1}=2.8 x(1-x), x_{0}=0.1$ ..... 32
$1.21<\mu<3, \bar{x}_{2}$ is asymptotically stable. ..... 32
1.3 Solution of $x_{n+1}=3.55 x(1-x), x_{0}=0.1$ ..... 33
$1.4 \mu>3, \bar{x}_{2}$ is unstable. ..... 33
5.1 Plot of $y_{n+1}=\frac{1+y_{n}+2 y_{n-2}}{y_{n}+2 y_{n-2}}$. ..... 94
5.2 Plot of $y_{n+1}=\frac{2+y_{n}+3 y_{n-3}}{y_{n}+4 y_{n-3}}$. ..... 96
5.3 Plot of $y_{n+1}=\frac{2}{y_{n}}+\frac{3}{y_{n-2}}$ ..... 96
5.4 Plot of $y_{n+1}=\frac{3+y_{n}+7 y_{n-3}}{y_{n}+0.4 y_{n-3}}$ ..... 99
5.5 Plot of $y_{n+1}=\frac{5}{y_{n-2}}$ ..... 99
5.6 Phase state graph of $y_{n+1}=\frac{2+y_{n}+4 y_{n-3}}{y_{n}+4 y_{n-3}}$ ..... 102
5.7 Phase state graph of $y_{n+1}=\frac{3+y_{n}+7 y_{n-3}}{y_{n}+0.4 y_{n-3}}$ ..... 102
5.8 Time series solution $y_{n+1}=\frac{3+y_{n}+7 y_{n-3}}{y_{n}+0.4 y_{n-3}}$ ..... 103


#### Abstract

In this thesis, we investigate the periodic character, invariant intervals,oscillation and global stability of all positive solutions of the equation : $$
x_{n+1}=\frac{\alpha+\beta x_{n}+\gamma x_{n-k}}{B x_{n}+C x_{n-k}}
$$ where the parameters, $\alpha, \beta, \gamma, \mathrm{B}$, and C and the initial conditions are nonnegative. We give a detailed description of the semicyles of solutions, and determine conditions that the equilibrium points are globally asymptotically stable.

In particular, our monograph is a generalization to the rational difference equation that was investigated in [6].


## Acknowledgements

I would like to thank Professor Mohammad Saleh, my supervisor, for his many suggestions and constant support during this research. I am also thankful to him for his guidance through the early years of chaos and confusion. He shared with me his knowledge of convex analysis and provided many useful references and friendly encouragement. He is a wonderful person. His friendly encouragement was crucial to the successful completion of this project.

I would like to thank Dr. Wasfi Alkafri for his many suggestions and guidance. I would also like to thank Dr. Hasan Yousef.

Of course, I am so grateful to my parents for their patience and love. Without them this work would never have come into existence. A special thanks to my family. I would also like to thank the Friends of mine .

Finally, I am so grateful to Birzeit University for its important role in all aspects: Economical, Social, and Political to our country.

Palestine, Ramallah
June 7, 2007

## Introduction

Difference equations are relatively new discipline within the fields of science and engineering. Difference equations appear in the literature under variety of names. There are large number of applications on dynamical systems and difference equations. These applications include mathematical, physical, biological, economical, and social science. Rational difference equations lack of complete theory, and the study of such equations is quite challenging and still at its infancy.

Rational difference equations are of great importance in their own right. And furthermore that results about such equation offer prototypes towards the development of the basic theory of the global behavior of solutions of nonlinear difference equations of order greater than one. The techniques and results about these equations are also useful in analyzing the equations in the mathematical models of various biological systems and other applications.

The Dynamical characteristics and qualitative behavior of positive solutions of some higher order nonlinear difference equations have been investigated by many authors.

Dehgan and Douraki [3] investigated the global stability, invariant intervals, semicycles, and the boundness of the equation:

$$
\begin{equation*}
x_{n+1}=\frac{x_{n}+p}{x_{n}+q x_{n-k}}, \quad n=0,1,2, \cdots \tag{0.1}
\end{equation*}
$$

where the parameters $p$ and $q$ are nonnegative and the initial conditions $x_{-k}, \ldots, x_{0}$ are positive real numbers, $k=\{1,2,3, \cdots\}$.

Li and Sun in [16] investigated the periodic character, invariant intervals, oscillation, and global stability of all positive solutions of the equation

$$
\begin{equation*}
x_{n+1}=\frac{p x_{n}+x_{n-k}}{q+x_{n-k}}, \quad n=0,1,2, \cdots \tag{0.2}
\end{equation*}
$$

where the parameters $p$ and $q$ and the initial conditions $x_{-k}, x_{-k+1}, \cdots, x_{-1}, x_{0}$ are nonnegative real numbers, $k=\{1,2,3, \cdots\}$.
M. Saleh and M. Aloqeili in [15] investigated the equation

$$
\begin{equation*}
y_{n+1}=A+\frac{y_{n}}{y_{n-k}}, n=0,1,2, \ldots \tag{0.3}
\end{equation*}
$$

M. Saleh and M. Aloqeili in [14] and H.M. El-Owaidy, A.M. Ahmed, and M.S. Mousa [9] investigated the global asymptotic stability, periodicity and semi-cycle analysis of the unique positive equilibrium point of the equation

$$
\begin{equation*}
y_{n+1}=A+\frac{y_{n-k}}{y_{n}}, n=0,1,2, \ldots \tag{0.4}
\end{equation*}
$$

DeVault in [5] investigated the global stability and periodic character of solutions of the equation

$$
\begin{equation*}
x_{n+1}=\frac{p+x_{n-k}}{q x_{n}+x_{n-k}}, \quad n=0,1,2, \cdots \tag{0.5}
\end{equation*}
$$

where the parameters $p$ and $q$ and the initial conditions $x_{-k}, x_{-k+1}, \cdots, x_{-1}, x_{0}$ are nonnegative real numbers, $k=\{1,2,3, \cdots\}$.
M.J. Douraki, M. Dehghan, and M. Razzaghi in [10] and [4] investigated the qualitative behavior of the equations

$$
\begin{equation*}
x_{n+1}=\frac{p+q x_{n-k}}{1+x_{n}}, n=0,1,2, \ldots \tag{0.6}
\end{equation*}
$$

and

$$
\begin{equation*}
x_{n+1}=\frac{p+q y_{n}}{1+y_{n-k}}, n=0,1,2, \ldots \tag{0.7}
\end{equation*}
$$

S. Abu Bahaa in [1] has investigated the local and global stability, invariant intervals, semicycles, periodic character of solutions of the difference equation

$$
\begin{equation*}
x_{n+1}=\frac{\beta x_{n}+\gamma x_{n-k}}{B x_{n}+C x_{n-k}}, \quad n=0,1,2, \cdots \tag{0.8}
\end{equation*}
$$

where the parameters $\beta, \gamma, \mathrm{B}$, and C and the initial conditions $x_{-k}, x_{-k+1}, \cdots, x_{-1}, x_{0}$ are nonnegative real numbers, $k=\{1,2,3, \cdots\}$.

To analyze equation $x_{n+1}=\frac{\alpha+\beta x_{n}+\gamma x_{n-k}}{B x_{n}+C x_{n-k}}$ theoritically, it is a good idea to study the difference equation

$$
x_{n+1}=\frac{\alpha+\beta x_{n-1}+\gamma x_{n-1}}{B x_{n}+C x_{n-1}}, n=0,1,2, \cdots
$$

where the parameters $\alpha, \beta, \gamma, \mathrm{B}$, and C are nonnegative real numbers and the initial conditions $x_{-1}, x_{0}$ are arbitrary positive real numbers.

The goal of our research on rational difference equations is to determine the character of solutions of equation

$$
x_{n+1}=\frac{\alpha+\beta x_{n}+\gamma x_{n-k}}{B x_{n}+C x_{n-k}}, n=0,1,2, \ldots
$$

for all nonnegative parameters $\alpha, \beta, \gamma, \mathrm{B}$ and C and nonnegative initial conditions $x_{-k}, x_{-k+1}, \ldots, x_{0}$. We are particularly interested in the asymptotic behavior of solutions, that is, the behavior of the solution as $n \rightarrow \infty$. We will determine the conditions for stability and give detailed description for Invariant interval, Existence of two-period solution, and Semicyle analysis.

Chapter 1 is an introduction to Difference equations. It includes linear and nonlinear first order difference equation or one dimensional maps on the real line, kth order Difference equations, and Equilibrium point concept. we give solution methods for linear difference equations of any order, and complete analysis of stability for many famous equations such as Linear Difference equations and Logistic Map. It includes Cobweb diagram, an effective graphical iteration methods to determine the stability of fixed points.

Chapter 2 introduces Rational Difference Equation, and some definitions and theorems that will be used next.

In Chapter 3 we investigate the rational difference equation

$$
x_{n+1}=\frac{\alpha+\beta x_{n}+\gamma x_{n-k}}{B x_{n}+C x_{n-k}}, n=0,1, \ldots
$$

We do change of variable to reduce number of parameters. Then we find the equilibrium point, and determine the conditions for stability. We give a detailed description of invariant intervals. Then we determine the conditions to Existence of two-cycles and semicycles. It is important to mention that chapter 3 has been done independently with Aseel Farhat.

In Chapter 4 we examine the character of solution of

$$
x_{n+1}=\frac{\alpha+\beta x_{n}+\gamma x_{n-k}}{B x_{n}+C x_{n-k}}, n=0,1, \ldots
$$

when one or more of the parameters are zero.
Finally, Chapter 5 presents numerical solutions obtained by using computer which is very good. We use a powerful Matlab and create mfiles to get plots and numerical solutions of equations. We also create Phase Space Diagram which is one of the best graphical methods to illustrate the various notions of stability. We compare between theoretical approach and computational approach, this is an important part of my thesis.

## Chapter 1

## Preliminary and Basic Theory Of Difference Equation

### 1.1 Introduction

The theory of dynamical systems is a major mathematical discipline closely intertwined with most of the main areas of mathematics. Its mathematical core is the study of the global orbit structure of maps and flows with emphasis on properties invariant under coordinate changes. Its concepts, methods, and paradigms greatly stimulate research in many sciences and have given rise to the vast new area of applied dynamics (also called nonlinear science or chaos theory). Although the field of dynamical systems comprises several major disciplines, we are interested mainly in dynamics of difference equations. The theory of dynamical systems is inseparable connected with several other areas, primarily difference equations and differential equations.

The dynamic of any situation refers to how the situation changes over the course
of time. A dynamical system is a physical setting with rules for how the setting changes or evolves from one moment of time to the next, i.e. a dynamical system is a system that changes over time. [1]. Dynamical system is contrast to static system which does not change over time.

When we model a system, we usually idealize the system in term of its state variable of the system, which are quantities that represent the system itself. For example, moving body may be represented by state variable of velocity and position over time. Model of population dynamic, the system state variable me be the number of population that born, migrate, and dead and the existing population.

In other words, dynamical systems is the study of phenomena that evolve in space and / or time by looking at the dynamic behavior or the geometrical and topological properties of the solution. Whether a particular system comes from Economics, Biology, Physics, Chemistry, or even Social sciences, the dynamical systems is the subject that provides the mathematical tools for its analysis.

Now, we introduce the Dynamical system in point of view of mathematics. $A$ dynamical system is a system whose behavior at given time depends, in some sense, on its behavior at one or more previous times.the words "in some sense" in the preceding sentence should be taken to mean that we may or may not have a clue as to how current state of a system depends on a past state; but we have reason to believe that it does. Furthermore, it is the task of the mathematical modeler to come up with a
mathematical construct, a model that will describe this relationship between current and past states of the system so that predictions about the future course of events for the system may be made with some degree of accuracy. [1].

### 1.2 Difference Equations

Dynamical systems has appeared in mathematics and engineering in many different forms and names regardless that they lead to same discipline. Our particular system is the system whose state depends on input history. In discrete time system, we call such system is difference equation which is equivalent to differential equation in continuous time. In this section we will talk about difference equation: definition, solution, difference equations in literature, and disciplines. While the behavior of solution of difference equation is left and we will discuss in chapter two. Difference equation is an equation involving differences. In this research We will investigate difference equation from two points of view: as sequence of numbers, and iterated function. they are equivalent, but we look at them in different points of view and for different purposes.

1. A difference equation is a sequence of numbers that is generated recursively using a rule to relate each number in the sequence to previous numbers in the sequence. [1]

Example 1.1. (Fibonacci sequence)

The sequence $\{1,1,2,3,5,8,13,21,34, \ldots\}$ is called Fibonacci sequence, which is generated by the formula

$$
x_{k+2}=x_{k+1}+x_{k}
$$

where $x_{0}=x_{1}=1$ and $k=0,1,2, \cdots$.
2. Difference equation as an iterated map : Consider a map $f: \mathbb{R} \rightarrow \mathbb{R}$ where $R$ is the set of real numbers. Then the (positive) Orbit $O\left(x_{0}\right)$ of a point $x_{0} \in \mathbb{R}$ is defined to be the set of points

$$
O\left(x_{0}\right)=\left\{x_{0}, f\left(x_{0}\right), f^{2}\left(x_{0}\right), f^{3}\left(x_{0}\right), \cdots\right\}
$$

where $f^{2}=f \circ f, f^{3}=f \circ f \circ f$, etc. and $f \circ f\left(x_{0}\right)=f\left(f\left(x_{0}\right)\right)$
Example 1.2. (The Logistic Map) The following mathematical model may be of the form

$$
\begin{equation*}
y_{n+1}=\mu y_{n}-b y_{n}^{2} \tag{1.1}
\end{equation*}
$$

where $y_{n}$ be the size of a population of a certain species at time $n, \mu$ is the rate of growth of the population from one generation to another, and $b$ is the proportionality constant of interaction among numbers of the species. To simplify Equation( 1.1), we let $x_{n}=\frac{b}{\mu} y_{n}$. Hence,

$$
\begin{equation*}
x_{n+1}=\mu x_{n}\left(1-x_{n}\right) \tag{1.2}
\end{equation*}
$$

Equation(1.2) is called the logistic equation and the map $f(x)=\mu x(1-x)$ is called logistic map. by varying the value of $\mu$, this equation exhibits somewhat complicated dynamics.

In the remaining of this chapter we will discuss the methodology of solving Difference equations and investigate their solution as $n \rightarrow \infty$.

### 1.3 Solution of Linear first order Difference Equations

Definition 1.1. Consider a map $f: \Re \rightarrow \Re$. Let $x_{n}=f^{n}\left(x_{0}\right)$ where $x_{0} \in \Re$. The following equation

$$
\begin{equation*}
x_{n+1}=f\left(x_{n}\right) \tag{1.3}
\end{equation*}
$$

is called first order difference equation with initial value $x_{0}$.
Definition 1.2. A solution of difference equation is the set of numbers that makes the difference equation true for all values. In other words, by a solution of Eq.(1.3), we mean a sequence $\left\{x_{n}\right\}, \mathrm{n}=0,1,2, \ldots$, with $x_{n+1}=f\left(x_{n}\right)$ and given $x_{0}$, i.e., a sequence that satisfies the equation.

The nature of difference equations allows the solution to be calculated recursively. So it is easier and better to see the solution of the difference equation through algebraic formula. In this case the difference equation is called closed form.

The simplest maps to deal with are the linear maps and the simplest difference equations to solve are linear ones. Despite Linear equations play an important role in mathematics because are being used to illustrate many situations since their solutions are simple to achieve. Many cases in natural and social science are modeled by linear equations. We can find out the solution of linear first order difference equation by forward iteration with initial condition $x_{0}$. Let us consider the following difference equation

$$
x_{n+1}=a x_{n}
$$

with initial condition $x_{0}$. Observe that the equilibrium point $\bar{x}=0$. We get the solution by forward iteration with initial condition, $x_{0}$

$$
\begin{aligned}
& x_{1}=a x_{0} \\
& x_{2}=a x_{1}=a\left(a x_{0}\right)=a^{2} x_{0} \\
& x_{3}=a x_{2}=a^{3} x_{0} \\
& \vdots \\
& x_{n}=a^{n} x_{0}
\end{aligned}
$$

We can make the following results about the limiting behavior of the solution of equation $x_{n+1}=a x_{n}$ :

1. If $|a|<1$, then $\lim _{n \rightarrow \infty} x_{n}=0$
2. If $|a|>1$, then $\lim _{n \rightarrow \infty} x_{n}=\infty$
3. If $a=1$, then every point is an equilibrium point.
4. If $a=-1$, then $x_{n}=\left\{\begin{array}{ccc}x_{0} & \text { if } n \text { is even } \\ -x_{0} & \text { if } n \text { is odd }\end{array}\right.$ or $x_{n}=(-1)^{n} x_{0}$

Example 1.3. Assume we have the following difference equation

$$
x_{n+1}=a x_{n}+b
$$

with initial value $x_{0}$ and we have to solve this equation. There are three cases :

1. $a \neq 1$ Observe that the equilibrium point

$$
\bar{x}=\frac{b}{1-a}
$$

The solution of difference equation can be calculated recursively

$$
\begin{align*}
& x_{1}=a x_{0}+b \\
& x_{2}=a x_{1}+b=a\left(a x_{0}+b\right)+b=a^{2} x_{0}+a b+b \\
& x_{3}=a x_{2}+b=a\left(a^{2} x_{0}+a b+b\right)+b=a^{3} x_{0}+a^{2} b+a b+b \\
& x_{4}=a x_{3}=a\left(a^{3} x_{0}+a^{2} b+a b+b\right)+b=a^{4} x_{0}+a^{3} b+a^{2} b+a b+b \\
& \vdots \\
& x_{n}=a^{n} x_{0}+a^{n-1} b+a^{n-2} b+\cdots+a b+b \\
& x_{n}=a^{n} x_{0}+b\left(a^{n-1}+a^{n-2}+\cdots+a+1\right) \\
& x_{n}=a^{n} x_{0}+b\left(\frac{1-a^{n}}{1-a}\right), a \neq 1 \\
& \quad x_{n}=\left(x_{0}+\frac{b}{a-1}\right) a^{n}+\frac{b}{1-a}, a \neq 1 \tag{1.4}
\end{align*}
$$

Using the formula of Eq.(1.4), the following conclusions can be easily verified:
(a) If $|a|<1$, then $\lim _{n \rightarrow \infty} x_{n}=\frac{b}{1-a}=\bar{x}$
(b) If $|a|>1$, then $\lim _{n \rightarrow \infty} x_{n}= \pm \infty$, depending on weather $x_{0}+\frac{b}{1-a}$ is positive or negative, respectively.
2. If $a=1$, then $x_{n}=x_{0}+n b$ and $\lim _{n \rightarrow \infty} x_{n}= \pm \infty$
3. If $a=-1$, then $x_{n}=(-1)^{n} x_{0}+\left\{\begin{array}{cc}0 ; & \text { if } n \text { is even } \\ b ; & \text { if } n \text { is odd }\end{array}\right.$

### 1.4 Solutions of Difference Equations of Higher Order

The normal form of $k^{t h}$ order nonhomogeneous linear difference equation is given by:

$$
\begin{equation*}
x_{n+k}+p_{1}(n) x_{n+k-1}+p_{2}(n) x_{n+k-2}+\cdots+p_{k}(n) x_{n}=g(n) \tag{1.5}
\end{equation*}
$$

where $p_{i}(n)$ and $g(n)$ are real valued functions defined for $n \geq n_{0}$ and $p_{k}(n) \neq 0$. If $g(n)=0$, then the Eq. (1.5) is said to be a homogeneous equation. Now the equation:

$$
\begin{equation*}
x_{n+k}+p_{1} x_{n+k-1}+p_{2} x_{n+k-2}+\cdots+p_{k} x_{n}=0 \tag{1.6}
\end{equation*}
$$

is called linear difference equation of $k^{t h}$ order with constant coefficients. To the end of this section we will give all possible solutions of Eq.( 1.6), and the solutions of Eq.( 1.5) have been investigated in [7]

### 1.4.1 Solutions of $k^{\text {th }}$ order homogeneous linear difference with constant coefficients

Now, consider the $k^{\text {th }}$ order homogeneous linear difference equation (1.6) where the $p_{i}$ 's are constant and $p_{k} \neq 0$. Define $\lambda$ to be the characteristic root of Eq.(1.6), then $\lambda^{n}$ is a solution of Eq.( 1.6). Substitute $\lambda^{n}$ into Eq.( 1.6), we obtain:

$$
\begin{equation*}
\lambda^{n}+p_{1} \lambda^{k-1}+\cdots+p_{1}=0 \tag{1.7}
\end{equation*}
$$

which is called the characteristic equation of Eq.(1.6).
The general solution of Eq.( 1.6) has different forms depending on $\lambda^{s}$.

1. Distinct roots

Suppose that the characteristic roots $\lambda_{1}, \lambda_{2}, \cdots, \lambda_{k}$ are distinct. i.e.

$$
\left|\lambda_{1}\right| \neq\left|\lambda_{2}\right| \neq \cdots \neq\left|\lambda_{k}\right|
$$

So the general solution is:

$$
x_{n}=c_{1} \lambda_{1}^{n}+c_{2} \lambda_{2}^{n}+\cdots+c_{k} \lambda_{k}^{n}
$$

Example 1.4. Find the solution of the following difference equation

$$
x_{n+2}+2 x_{n+1}-8 x_{n}=0, x_{0}=2, x_{1}=3
$$

Solution:

The characteristic equation of the above difference equation is:

$$
\lambda^{2}+2 \lambda-8=0
$$

The characteristic roots are: $\lambda_{1}=2, \lambda_{2}=-4$, The general solution is given by

$$
\begin{aligned}
& x_{n}=c_{1}(2)^{n}+c_{2}(-4)^{n} \\
& x_{0}=2=c_{1}+c_{2} \\
& x_{1}=3=2 c_{1}-4 c_{2}
\end{aligned}
$$

Thus $c_{1}=\frac{11}{6}$ and $c_{2}=\frac{1}{6}$. Consequently, the general solution is:

$$
x_{n}=\frac{11}{6}(2)^{n}+\frac{1}{6}(-4)^{n}
$$

2. Repeated Roots

$$
\lambda_{1}=\lambda_{2}=\cdots=\lambda_{m}=\lambda, 2 \leq m \leq k
$$

so the general solution of difference equation( 1.6) is given by:

$$
x_{n}=c_{1} \lambda^{n}+c_{2} n \lambda^{n}+\cdots+c_{m} n^{m-1} \lambda^{n}+c_{m+1} \lambda_{m+1}^{n}+\cdots+c_{k} \lambda_{k}^{n}
$$

Example 1.5. Find the solution of the following difference equation

$$
x_{n+2}+6 x_{n+1}+9 x_{n}=0, x_{0}=1, x_{1}=0
$$

## Solution:

The characteristic equation of the above difference equation is:

$$
\lambda^{2}+6 \lambda+9=0
$$

so $\lambda_{1}=\lambda_{2}=-3$, The general solution is given by

$$
\begin{aligned}
& x_{n}=c_{1}(-3)^{n}+c_{2} n(-3)^{n} \\
& x_{0}=1=c_{1} \\
& x_{1}=0=-3 c_{1}-3 c_{2}
\end{aligned}
$$

Thus, $c_{2}=-1$ and, consequently,

$$
\begin{aligned}
x_{n} & =(-3)^{n}-n(-3)^{n} \\
& =(-3)^{n}(1-n)
\end{aligned}
$$

3. The absolute value of the roots are equal
i.e.

$$
\left|\lambda_{1}\right|=\left|\lambda_{2}\right|=\cdots=\left|\lambda_{k}\right|
$$

- The characteristic roots are equal the general solution is:

$$
\begin{aligned}
x_{n} & =c_{1} \lambda^{n}+c_{2} n \lambda^{n}+\cdots+c_{k} n^{k-1} \lambda^{n} \\
& =\left(c_{1}+c_{2} n+\cdots+c_{k} n^{k-1}\right) \lambda^{n}
\end{aligned}
$$

- The characteristic roots are not equal

$$
\lambda_{1}=\lambda_{2}=\cdots=\lambda_{m}=\lambda
$$

and

$$
\lambda_{m+1}=\lambda_{m+2}=\cdots=\lambda_{k}=-\lambda
$$

The general solution is given by:

$$
\begin{aligned}
& x_{n}=\left(c_{1}+c_{2} n+c_{3} n^{2}+\cdots+c_{m} n^{m-1}\right) \lambda^{n}+ \\
& \left(c_{m+1}+c_{m+2} n+c_{m+3} n^{2}+\cdots+c_{k} n^{k-m-1}\right)(-1)^{n} \lambda^{n}
\end{aligned}
$$

Example 1.6. Find the solution of the following difference equation

$$
x_{n+2}-4 x_{n}=0
$$

Solution: The characteristic equation is

$$
\begin{gathered}
\lambda^{2}-4=0 \\
\lambda= \pm 2
\end{gathered}
$$

So the general solution is given by:

$$
\begin{aligned}
x_{n} & =c_{1} 2^{n}+c_{2}(-2)^{n} \\
& =c_{1} 2^{n}+c_{2}(-1)^{n} 2^{n} \\
& =\left(c_{1}+(-1)^{n} c_{2}\right) 2^{n}
\end{aligned}
$$

4. Some of roots are complex

Assume that

$$
\lambda_{1}=\alpha+i \beta
$$

and

$$
\lambda_{2}=\alpha-i \beta
$$

and that $\lambda_{3}, \lambda_{4}, \cdots, \lambda_{k}$ are all real and distinct such that

$$
\left|\lambda_{3}\right|>\left|\lambda_{4}\right|>\cdots>\left|\lambda_{k}\right|
$$

where

$$
\begin{aligned}
\lambda_{1} & =\alpha+i \beta \\
& =r e^{i \phi} \\
& =r(\cos \phi+i \sin \phi)
\end{aligned}
$$

and

$$
\begin{aligned}
\lambda_{2} & =\alpha-i \beta \\
& =r e^{-i \phi} \\
& =r(\cos \phi-i \sin \phi)
\end{aligned}
$$

Then the general solution of Eq.(1.6) is given by:

$$
\begin{aligned}
x_{n} & =c_{1} r^{n} e^{i n \phi}+c_{2} r^{n} e^{-i n \phi}+c_{3} \lambda_{3}^{n}+\cdots+c_{k} \lambda_{k}^{n} \\
& =c_{1} r^{n}(\cos n \phi+i \sin n \phi)+c_{2} r^{n}(\cos n \phi-i \sin n \phi)+c_{3} \lambda_{3}^{n}+\cdots+c_{k} \lambda_{k}^{n} \\
& =\left(c_{1}+c_{2}\right) r^{n} \cos n \phi+\left(c_{1}-c_{2}\right) r^{n} i \sin n \phi+c_{3} \lambda_{3}^{n}+\cdots+c_{k} \lambda_{k}^{n} \\
& =r^{n}\left[\left(c_{1}+c_{2}\right) \cos n \phi+\left(c_{1}-c_{2}\right) i \sin n \phi\right]+c_{3} \lambda_{3}^{n}+\cdots+c_{k} \lambda_{k}^{n} \\
& =r^{n}\left[a_{1} \cos n \phi+a_{2} \sin n \phi\right]+c_{3} \lambda_{3}^{n}+\cdots+c_{k} \lambda_{k}^{n}
\end{aligned}
$$

where $a_{1}=c_{1}+c_{2}$ and $a_{2}=\left(c_{1}-c_{2}\right) i$. Now, Let

$$
\cos \omega=\frac{a_{1}}{\sqrt{a_{1}^{2}+a_{2}^{2}}}, \sin \omega=\frac{a_{2}}{\sqrt{a_{1}^{2}+a_{2}^{2}}}, \omega=\arctan \left(\frac{a_{2}}{a_{1}}\right)
$$

The solution will be

$$
\begin{aligned}
x_{n} & =r^{n} \sqrt{a_{1}^{2}+a_{2}^{2}}[\cos \omega \cos n \phi+\sin \omega \sin n \phi]+c_{3} \lambda_{3}^{n}+\cdots+c_{k} \lambda_{k}^{n} \\
& =r^{n} \sqrt{a_{1}^{2}+a_{2}^{2}} \cos (n \phi-\omega)+c_{3} \lambda_{3}^{n}+\cdots+c_{k} \lambda_{k}^{n} \\
& =A r^{n} \cos (n \phi-\omega)+c_{3} \lambda_{3}^{n}+\cdots+c_{k} \lambda_{k}^{n}
\end{aligned}
$$

where

$$
\begin{aligned}
A & =\sqrt{a_{1}^{2}+a_{2}^{2}} \\
r & =\sqrt{\alpha^{2}+\beta^{2}} \\
\phi & =\arctan \left(\frac{\beta}{\alpha}\right)
\end{aligned}
$$

Example 1.7. Solve the difference equation

$$
x_{n+3}-4 x_{n+2}+6 x_{n+1}-4 x_{n}=0
$$

Solution:

The characteristic equation is

$$
\begin{gathered}
\lambda^{3}-4 \lambda^{2}+6 \lambda-4=0 \\
(\lambda-2)\left(\lambda^{2}-2 \lambda+2\right)=0
\end{gathered}
$$

The characteristic roots are: $\lambda=2, \lambda=1+i$, and $\lambda=1-i$. Therefore, the general solution is

$$
x_{n}=c_{1} 2^{n}+A(\sqrt{2})^{n} \cos \left(n \frac{\pi}{4}-\omega\right)
$$

### 1.4.2 Solutions of $k^{\text {th }}$ order nonhomogeneous linear difference with constant coefficients

The main idea of solving such difference equations is to find particular solution in addition to homogeneous solution, and there are some techniques discussed in this manner in [7].
Example 1.8. Find the general solution of

$$
x_{n+2}-3 x_{n+1}+2 x_{n}=4^{n}-n^{2}
$$

Solution:
Let $x_{0}, x_{1}$ be two initial conditions. Then

$$
x_{n}=x_{h n}+x_{p n}
$$

where $x_{n}$ is the general solution.
$x_{h n}$ is the homogeneous solution.
$x_{p n}$ is the particular solution.
To find the homogeneous solution: solve the characteristic equation:

$$
\begin{gathered}
r^{2}-3 r+2=0 \\
\Rightarrow \quad r^{2}-3 r+2=(r-1)(r-2)=0 \\
\\
r_{1}=1, r_{2}=2
\end{gathered}
$$

Then, the homogeneous solution is:

$$
\begin{aligned}
x_{h n} & =a r_{1}^{n}+b r_{2}^{n} \\
& =a+b 2^{n}
\end{aligned}
$$

To find particular solution, let $x_{p n}=c 4^{n}+d n^{2}+e n+f$
substituting this potential solution into the equation and equating coefficients as
following

$$
\begin{aligned}
& x_{p n}=c 4^{n}+d n^{2}+e n+f \\
& x_{p n+1}=c 4^{n+1}+d(n+1)^{2}+e(n+1)+f \\
& x_{p n+2}=c 4^{n+2}+d(n+2)^{2}+e(n+2)+f
\end{aligned}
$$

Hence, we get
$c 4^{n+2}+d(n+2)^{2}+e(n+2)+f-3\left(c 4^{n+1}+d(n+1)^{2}+e(n+1)+f\right)+2 c 4^{n}+d n^{2}+e n+f=4^{n}-n^{2}$
after doing simple algebraic calculations, we get

$$
\begin{aligned}
& \qquad 6 c 4^{n}-2 d n+d-e=4^{n}-n^{2} \\
& \Rightarrow 6 c=1 \Rightarrow c=\frac{1}{6} \\
& \Rightarrow-2 d=0 \Rightarrow d=0 \\
& \Rightarrow d-e=0 \Rightarrow e=0
\end{aligned}
$$

Thus, the general solution of the equation is:

$$
x_{n}=a+b(2)^{n}+\frac{1}{6} 4^{n}
$$

To find the values of constants $a$ and $b$ the initial conditions $x_{0}, x_{1}$ must be provided.

### 1.5 Solution of Nonlinear Difference Equations

In fact, most of Difference Equations arise from real applications are nonlinear. And most nonlinear difference equations cannot be solved explicitly. However, some of the nonlinear difference equations can be transformed into linear difference equations by change of variable techniques [7].

In this section we introduce a few types of linear transformation techniques.

Type 1. Equations of Riccati type

$$
\begin{equation*}
x_{n+1} x_{n}+p(n) x_{n+1}+q(n) x_{n}=0 \tag{1.8}
\end{equation*}
$$

The change of variable $z_{n}=\frac{1}{x_{n}}$ transform the Riccati equation (1.8) to the linear difference equation

$$
\begin{equation*}
q(n) z_{n+1}+p(n) z_{n}+1=0 \tag{1.9}
\end{equation*}
$$

The nonhomogeneous equation of Riccati type

$$
\begin{equation*}
x_{n+1} x_{n}+p(n) x_{n+1}+q(n) x_{n}=g(n) \tag{1.10}
\end{equation*}
$$

requires a different transformation. Let $y_{n}=\frac{z_{n+1}}{z_{n}}-p(n)$ and substitute it in Eq.(1.10) to get

$$
\begin{equation*}
z_{n+2}+(q(n)-p(n+1)) z_{n+1}-(g(n)+p(n) q(n)) z_{n}=0 \tag{1.11}
\end{equation*}
$$

Example 1.9. Solve the difference equation

$$
x_{n+1} x_{n}-x_{n+1}+x_{n}=0
$$

Solution:

The equation is Riccati type and we can solve it By letting $x_{n}=\frac{1}{z_{n}}$. This gives us the equation

$$
z_{n+1}=z_{n}-1
$$

which is first order linear difference equation whose solution is given by

$$
z_{n}=c-n
$$

therefore,

$$
x_{n}=\frac{1}{c-n}
$$

Type 2. Equations of general Riccati type:

$$
\begin{equation*}
x_{n+1}=\frac{a(n) x_{n}+b(n)}{c(n) x_{n}+d(n)} \tag{1.12}
\end{equation*}
$$

such that $c(n) \neq 0, a(n) d(n)-b(n) c(n) \neq 0$ for all $n \geq 0$.

Let

$$
c(n) x_{n}+d(n)=\frac{y_{n+1}}{y_{n}}
$$

then

$$
x_{n}=\frac{y_{n+1}}{c(n) y_{n}}-\frac{d(n)}{c(n)}
$$

Substitute it in (1.12) to obtain

$$
\frac{y_{n+2}}{c(n+1) y_{n+1}}-\frac{d(n+1)}{c(n+1)}=\frac{a(n)+\left[\frac{y_{n+1}}{c(n) y_{n}}-\frac{d(n)}{c(n)}\right]+b(n)}{\frac{y_{n+1}}{y_{n}}}
$$

By simplifying the above equation, we get

$$
\begin{gather*}
y_{n+2}+p_{1}(n) y_{n+1}+p_{2}(n) y_{n}=0,  \tag{1.13}\\
\left.y_{( } 0\right)=1, y_{1}=c(0) x_{0}+d(0)
\end{gather*}
$$

Where

$$
\begin{aligned}
& p_{1}(n)=-\frac{c(n) d(n+1)+a(n) c(n+1)}{c(n)} \\
& p_{2}(n)=(a(n) d(n)-b(n) c(n)) \frac{c(n+1)}{c(n)}
\end{aligned}
$$

Example 1.10. Solve the difference equation

$$
x_{n+1}=\frac{2 x_{n}+3}{3 x_{n}+2}
$$

Solution:

Here $a=2, b=3, c=3$, and $d=2$. Hence $a d-b c \neq 0$. By using the transformation

$$
\begin{equation*}
3 x_{n}+2=\frac{y_{n+1}}{y_{n}} \tag{1.14}
\end{equation*}
$$

we obtain the following homogeneous linear difference equation

$$
y_{n+2}-4 y_{n+1}-5 y_{n}=0, \quad y_{0}=1, \quad y_{1}=3 x_{0}+2
$$

And its characteristic equation is

$$
\lambda^{2}-4 \lambda-5=0
$$

hence, the characteristic roots $\lambda_{1}=5, \lambda_{2}=-1$.

## Hence

$$
y_{n}=c_{1} 5^{n}+c_{2}(-1)^{n} .
$$

By using formula (1.14), we have

$$
\begin{aligned}
x_{n} & =\frac{1}{3} \frac{y_{n+1}}{y_{n}}-\frac{2}{3} \\
& =\frac{1}{3} \frac{c_{1} 5^{n+1}+c_{2}(-1)^{n+1}}{c_{1} 5^{n}+c_{2}(-1)^{n}}-\frac{1}{3} \\
& =\frac{c_{1} 5^{n}-c_{2}(-1)^{n}}{c_{1} 5^{n}+c_{2}(-1)^{n}} \\
& =\frac{5^{n}-c(-1)^{n}}{5^{n}+c(-1)^{n}},
\end{aligned}
$$

where $c=\frac{c_{1}}{c_{2}}$.

Type 3. Homogeneous difference equation of the type

$$
\begin{equation*}
f\left(\frac{x_{n+1}}{x_{n}}, n\right)=0 \tag{1.15}
\end{equation*}
$$

Use the transformation $z_{n}=\frac{x_{n+1}}{x_{n}}$ to convert such an equation to a linear in $z_{n}$, which is can be solved easily.

Example 1.11. Solve the following difference equation

$$
\begin{equation*}
x_{n+1}^{2}-3 x_{n+1} x_{n}+2 x_{n}^{2}=0 \tag{1.16}
\end{equation*}
$$

Solution: By dividing over $x_{n}^{2}$, the equation (1.16) will be

$$
\left(\frac{x_{n+1}}{x_{n}}\right)^{2}-3\left(\frac{x_{n+1}}{x_{n}}\right)+2=0
$$

By letting

$$
z_{n}=\frac{x_{n+1}}{x_{n}}
$$

we get the following equation

$$
z_{n}^{2}-3 z_{n}+2=0
$$

and the last equation can be broken by factorizing to

$$
\left(z_{n}-2\right)\left(z_{n}-1\right)=0
$$

Thus, either

$$
z_{n}=2
$$

or

$$
z_{n}=1
$$

Hence, we get the following solution:

$$
x_{n+1}=2 x_{n}
$$

or

$$
x_{n+1}=x_{n}
$$

consequently

$$
x_{n}=2^{n} x_{0}
$$

or

$$
x_{n}=x_{0}
$$

Type 4. Consider the difference equation of the form

$$
\begin{equation*}
\left(\left(x_{n+k}\right)^{r_{1}}\right)\left(\left(x_{n+k-1}\right)^{r_{2}}\right) \ldots\left(\left(x_{n}\right)^{r_{k+1}}\right)=g(n) \tag{1.17}
\end{equation*}
$$

The change of variable $z_{n}=\ln x(n)$ transform Eq.(1.17) to

$$
\begin{equation*}
r_{1} z_{n+k}+r_{2} z_{n+k-1}+\ldots+r_{k+1} z_{n}=\ln g(n) \tag{1.18}
\end{equation*}
$$

Example 1.12. Solve the difference equation

$$
\begin{equation*}
x_{n+2}=\frac{x_{n+1}^{2}}{x_{n}^{2}} \tag{1.19}
\end{equation*}
$$

Solution:

Let $z_{n}=\ln x_{n}$, then substitute in $x_{n}=e^{z_{n}}$ Eq.(1.19) we obtain

$$
z_{n+2}-2 z_{n+1}+2 z_{n}=0
$$

which is second order linear difference equation and its characteristic equation is

$$
\lambda^{2}-2 \lambda+2=0
$$

The characteristic roots are $\lambda_{1}=1+i, \lambda_{2}=1-i$. Thus

$$
z_{n}=(\sqrt{2})^{n}\left[c_{1} \cos \left(\frac{n \pi}{4}\right)+c_{2} \sin \left(\frac{n \pi}{4}\right)\right] .
$$

Therefore,

$$
x_{n}=\exp \left[(\sqrt{2})^{n}\left\{c_{1} \cos \left(\frac{n \pi}{4}\right)+c_{2} \sin \left(\frac{n \pi}{4}\right)\right\}\right]
$$

### 1.6 Behavior of Solutions of Difference Equations

In this section we will try to determine the behavior of solution of difference equations in view of theoretical and computational approach. Moreover the difference equations have a complete theory in one dimension, so we will list all definitions and theorems with illustration examples. These examples have been chosen to help the reader to understand the notions and terminologies that have been used in next chapters. For this purpose we concentrate our investigation to the first order difference equations. As we mentioned we are particularly interested in the asymptotic behavior of solutions, that is, the behavior of the solution as $n \rightarrow \infty$. However, our research only looks at simple models, which can be easily solved analytically. This approach has two advantages: first, most of us are familiar with these models and can obtain their analytical or exact solutions in addition to numerical solutions for these models which can be obtained using Matlab and Maple. Second, the comparison between analytical and numerical results help us understand the power and the limits of numerical solutions.

Consider a map $f: \Re \rightarrow \Re$ where $\Re$ is the set of real numbers. Then the ( positive) orbit $O\left(x_{0}\right)$ of point $x_{0} \in \Re$ is defined to be the set of points

$$
O\left(x_{0}\right)=\left\{x_{0}, f\left(x_{0}\right), f^{2}\left(x_{0}\right), f^{3}\left(x_{0}\right), \cdots\right\}
$$

### 1.6.1 Equilibrium Points of Difference Equations

Let us consider the difference equation

$$
\begin{equation*}
x_{n+1}=f\left(x_{n}\right) \tag{1.20}
\end{equation*}
$$

Definition 1.3. [8] A point $\bar{x}$ is said to be a fixed point of the map $f$ or an equilibrium point of the Eq. (1.20) if $f(\bar{x})=\bar{x}$.

Example 1.13. Determine the fixed points of the following function

$$
f(x)=x^{2}-4 x+6
$$

Solution: We can find the fixed points by solving the following equation:

$$
f(x)=x
$$

then, we get

$$
x^{2}-4 x+6=x
$$

hence

$$
x^{2}-5 x+6=0
$$

then

$$
(x-2)(x-3)=0
$$

hence, there are two fixed points

$$
\bar{x}=2 \text { and } \bar{x}=3
$$

Example 1.14. Find the Equilibrium points of the following difference equation

$$
x_{n+1}=2 x_{n}\left(1-x_{n}\right)
$$

Solution: Set

$$
\bar{x}=2 \bar{x}(1-\bar{x})
$$

by solving the previous equation, we get two equilibrium points

$$
\bar{x}=0 \text { and } \bar{x}=\frac{1}{2}
$$

### 1.6.2 Stability Theory

One of the main objectives in the theory of dynamical systems is the study of the behavior of orbits near fixed points, in other words, the behavior of solutions of a difference equation near equilibrium points, such investigation is called Stability theory, which will be one of our main focus henceforh. To do this investigation, we begin by introducting the basic notions of stability.

Definition 1.4. [8] Let $f: I \longrightarrow I$ where $I$ is an interval in the set of real numbers $\Re$ and $\bar{x}$ be an equilibrium point of the difference equation

$$
\begin{equation*}
x_{n+1}=f\left(x_{n}\right) \tag{1.21}
\end{equation*}
$$

then

1. The equilibrium point $\bar{x}$ of Eq. 1.21 is called stable if for every $\epsilon$, there exists $\delta$ such that if

$$
\left|x_{0}-\bar{x}\right|<\delta
$$

then

$$
\left|x_{n}-\bar{x}\right|<\epsilon
$$

for all $\mathrm{n} \geq 1$, and all $x \in I$.
2. The equilibrium point $\bar{x}$ of Eq. 1.21 is called locally asymptotically stable or (asymptotically stable) if is it stable and if there exist $\gamma>0$ such that if

$$
\left|x_{0}-\bar{x}\right|<\gamma
$$

and

$$
\lim _{n \rightarrow \infty} x_{n}=\bar{x}
$$

3. The equilibrium point $\bar{x}$ of Eq. 1.21 is called global attractor if for every

$$
x_{0} \in I
$$

then

$$
\lim _{n \rightarrow \infty} x_{n}=\bar{x}
$$

4. The equilibrium point $\bar{x}$ of Eq. 1.21 is called global asymptotically stable (or globally stable)if it is stable and is global attractor.
5. the equilibrium point $\bar{x}$ of Eq. 1.21 is called unstable if it is not stable
6. the equilibrium point $\bar{x}$ of Eq. 2.1 is called repller if there exists $r>0$ such that if $x_{0} \in I$ and

$$
\left|x_{0}-\bar{x}\right|<r
$$

then there exists $N \geq 1$ such that

$$
\left|x_{N}-\bar{x}\right|>r
$$

Clearly, a repller is an unstable equilibrium point.

### 1.6.3 Graphical Iteration

One of the most effective graphical iteration methods to determine the stability of fixed points is Cobweb diagram on the $\left(x_{n}, x_{n+1}\right)$ or $\left(f^{n}\left(x_{0}\right), f^{n+1}\left(x_{0}\right)\right)$ plane. Cobweb diagrams provide a relatively quick way of representing the repeated application of an iterative function which are often used to simulate dynamics because iterative functions are complicated to predict the results, and studying the numerical results of applying the function again and again may not provide much insight into the long-term behavior of the dynamical system.

To accomplish cobweb diagram, we draw the curve $y=f(x)$ and the diagonal $y=x$ on the same plot.

We start at an initial point $x_{0}$. Then we move vertically until we hit the graph of $f$ at the point $\left(x_{0}, f\left(x_{0}\right)\right)$. We then travel horizontally to meet the line $y=x$ at the point $\left(f\left(x_{0}\right), f\left(x_{0}\right)\right)$. This determines $f\left(x_{0}\right)$ on the $x$ axis. to find $f^{2}\left(x_{0}\right)$, we move
again vertically until we hit the graph of $f$ at the point $\left(f\left(x_{0}\right), f^{2}\left(x_{0}\right)\right)$, and then we move horizontal to meet the line $y=x$ at the point $\left(f^{2}\left(x_{0}\right), f^{2}\left(x_{0}\right)\right)$. Continuing this process, we can evaluate all of the points in the orbit of $x_{0}$, namely, the set $\left\{x_{0}, f\left(x_{0}\right), f^{2}\left(x_{0}\right), \cdots, f^{n}\left(x_{0}\right), \cdots\right\}$ or equivalently $\left\{x_{0}, x_{1}, x_{2}, \cdots, x_{n}, \cdots\right\}$

Definition 1.5. Let $\mu>0$, then the difference equation

$$
\begin{equation*}
x_{n+1}=\mu x_{n}\left(1-x_{n}\right) \tag{1.22}
\end{equation*}
$$

is called discrete Logistic difference equation. And the function

$$
f_{\mu}(x)=\mu x(1-x)
$$

is called Logistic Map.
Example 1.15. Consider the difference equation $x_{n+1}=\mu x_{n}\left(1-x_{n}\right)$ for $\mu=2$ and $\mu=3.6$

## 1. Fined fixed points

2. Obtain numerical solution of the difference equation.
3. Determine the stability of fixed points by using Cobweb diagram.

Solution: To find the fixed points of $f_{\mu}$, we solve the equation $\mu x(1-x)=x$. This yields two equilibrium (fixed) points : $\bar{x}_{1}=0$ and $\bar{x}_{2}=\frac{\mu-1}{\mu}$.

- When $\mu=2.8$. The two fixed points are: $\bar{x}_{1}=0$ and $\bar{x}_{2}=0.6429$.

And the stability can be achieved from Cobweb diagram, see Fig.(1.2).

- When $\mu=3.55$. The two fixed points are: $\bar{x}_{1}=0$ and $\bar{x}_{2}=0.7183$. Observe that the solution of $x_{n+1}=3.55 x(1-x)$ does not converges, see Fig.(1.3). And from Cobweb diagram the equilibrium point $\bar{x}_{2}$ is unstable, see Fig.(1.4).

DiscreteLogistic-r=2.8


Figure 1.1: Solution of $x_{n+1}=2.8 x(1-x), x_{0}=0.1$


Figure 1.2: $1<\mu<3, \bar{x}_{2}$ is asymptotically stable.


Figure 1.3: Solution of $x_{n+1}=3.55 x(1-x), x_{0}=0.1$


Figure 1.4: $\mu>3, \bar{x}_{2}$ is unstable.

### 1.7 Criteria for Stability

In this section, we are going to introduce some powerful criteria for local stability of equilibrium(fixed) points. Equilibrium points are divided into two types: hyperbolic and non hyperbolic. A fixed point $\bar{x}$ of a map $f$ is said to be hyperbolic if $\left|f^{\prime}(\bar{x})\right| \neq 1$. Otherwise it is non hyperbolic.

Theorem 1.16. [8] (Criteria for Stability) Let $\bar{x}$ be a hyperbolic fixed point of a map $f$, where $f$ is continuously differentiable at $\bar{x}$. The following statements then holds true:

1. If $\left|f^{\prime}(\bar{x})\right|<1$, then the equilibrium point $\bar{x}$ of Eq. 1.21 is asymptotically stable.
2. If $\left|f^{\prime}(\bar{x})\right|>1$, then the equilibrium point $\bar{x}$ of Eq. 1.21 is un stable.

In Example 1.15, there are two fixed points :

$$
\bar{x}_{1}=0 \text { and } \bar{x}_{2}=\frac{\mu-1}{\mu}
$$

Observe that $f^{\prime}(x)=\mu(1-2 x)$

- $\bar{x}_{1}=0$. Thus $f^{\prime}(0)=\mu$, and hence $\bar{x}_{1}=0$ is stable when $0 \leq \mu<1$, and unstable when $\mu>1$
- $\bar{x}_{2}=\frac{\mu-1}{\mu}$. Thus $f^{\prime}\left(\bar{x}_{2}\right)=2-\mu$, and hence by theorem $1.16, \bar{x}_{2}$ is asymptotically stable if $|2-\mu|<1$. Solving the latter inequality for $\mu$, we obtain $1<\mu<3$. and $\bar{x}_{2}$ is unstable if $\mu<1$ and $\mu>3$. When $\mu=1, f^{\prime}\left(\bar{x}_{2}\right)=1$, and $\mu=3$, $f^{\prime}\left(\bar{x}_{2}\right)=-1$. These two cases will discuss next.

The stability criteria when $\bar{x}$ is non hyperbolic is summarized in the next two theorems. The following theorem treats the case when $f^{\prime}(\bar{x})=1$

Theorem 1.17. [8] Let $\bar{x}$ be a non hyperbolic fixed point $f^{\prime}(\bar{x})=1$ of a map $f$, where $f^{\prime}$ is continuous. The following statements then holds true:

- If $f^{\prime \prime}(\bar{x}) \neq 0$, then $\bar{x}$ is unstable.
- If $f^{\prime \prime}(\bar{x})=0$ and $f^{\prime \prime \prime}(\bar{x})>0$, then $\bar{x}$ is unstable.
- If $f^{\prime \prime}(\bar{x})=0$ and $f^{\prime \prime \prime}(\bar{x})<0$, then $\bar{x}$ is asymptotically stable.

The preceding theorem may be used to establish stability criteria for the case when $f^{\prime}(\bar{x})=-1$. But before doing so, we need to introduce the notion of Schwarzian derivative.

Definition 1.6. (The Schwarzian derivative). $S f$ of a function is gevin by

$$
S f=\frac{f^{\prime \prime \prime}(x)}{f^{\prime}(x)}-\frac{3}{2}\left[\frac{f^{\prime \prime}(x)}{f^{\prime}(x)}\right]^{2}
$$

Theorem 1.18. Let $\bar{x}$ be a fixed point of a map $f$ and $f^{\prime}(\bar{x})=-1$. If $f^{\prime \prime \prime}(\bar{x})$ is continuous, then the following statements hold:

- If $S f(\bar{x})<0$, then $\bar{x}$ is asymptotically stable.
- If $S f(\bar{x})>0$, then $\bar{x}$ is unstable.


## Chapter 2

## Preliminary and Basic Theory Of Rational Difference Equations

### 2.1 Rational Difference Equations

The general form for the rational difference equation is :

$$
x_{n+1}=\frac{a_{0}+a_{1} x_{1}+a_{2} x_{2}+\cdots+a_{k} x_{k}}{b_{0}+b_{1} x_{1}+b_{2} x_{2}+\cdots+b_{l} x_{l}}
$$

where the parameters $a_{0}, a_{1}, \cdots, a_{m}, b_{0}, \cdots, b_{m}$ are positive real numbers and the initial conditions $x_{1}, \cdots, x_{m}$ are nonnegative real numbers where $m=\max \{k, l\}$. The study of rational difference equations of order greater than one is quite challenging and rewarding, and the results about these equations offer prototypes towards the development of the basic theory of the global behavior of solutions of nonlinear difference equations of order greater than one. The techniques and results about these equations are also useful in analyzing the equations in the mathematical models of various biological systems and other applications.

The study of properties of rational difference equations has been an area of intense
interest in recent years and reference therein. [11]. Ladas and Kulenovic in [11] have discussed the dynamics of second order rational difference equations.In this research we will investigate the following $k^{\text {th }}$ order difference equation:

$$
x_{n+1}=\frac{\alpha+\beta x_{n}+\gamma x_{n-k}}{B x_{n}+C x_{n-k}}
$$

Solution of any difference equation depends on both parameters and initial conditions.
Solution of $K^{\text {th }}$ order rational difference equation may exhibit one or more of the following characteristics:

- The solution converges to an equilibrium point.
- The solution converges to aperiodic solution.
- The solution contain one or more unbounded subsequences.
- The solution is bounded but does not converge to an equilibrium point.
- Every solution is periodic with the same period.


### 2.2 Definitions

Here, we list some definitions which will be useful in our investigation.
Proposition 2.1. [3] Let I be some interval of real numbers and let

$$
f: I \times I \longrightarrow I
$$

be a continuous differentiable function. Then for every set of initial conditions $x_{-k}, \cdots, x_{-1}, x_{0}$ $I$, the difference equation

$$
\begin{equation*}
x_{n+1}=f\left(x_{n}, x_{n-k}\right), n=0,1, \cdots \tag{2.1}
\end{equation*}
$$

has a unique solution $\left\{x_{n}\right\}_{n=-k}^{\infty}$

Definition 2.1. [16] A point is $\bar{x}$ is called an equilibrium point of equation (2.1) if

$$
\bar{x}=f(\bar{x}, \bar{x})
$$

that is

$$
x_{n}=\bar{x}
$$

for $n \geq 0$ is a solution of equation (2.1), or equivalently, $\bar{x}$ is a fixed point of $f$.
Definition 2.2. (Periodicity)

1. A solution $\left\{x_{n}\right\}_{n=-k}^{\infty}$ of a difference equation is said to be periodic with period $p$ if $x_{n+p}=x_{n}$ for all $n \geq-k$.
2. A solution $\left\{x_{n}\right\}_{n=-k}^{\infty}$ of a difference equation is said to be periodic with prime period $p$ or $p$-cycle if it is periodic with period $p$ and $p$ is the least positive integer for which $x_{n+p}=x_{n}$ holds.

Definition 2.3. [16] Let $\bar{x}$ be an equilibrium point of Eq.(2.1), and assume that $I$ is an interval of real numbers. Then

1. The equilibrium point $\bar{x}$ of Eq. 2.1 is called stable if for every $\epsilon$, there exists $\delta$ such that if

$$
x_{-k}, \cdots x_{-1}, x_{0} \in I
$$

and

$$
\left|x_{-k}-\bar{x}\right|+\left|x_{-k+1}-\bar{x}\right|+\cdots+\left|x_{0}-\bar{x}\right|<\delta
$$

then

$$
\left|x_{n}-\bar{x}\right|<\epsilon
$$

for all $\mathrm{n} \geq-k$
2. The equilibrium point $\bar{x}$ of Eq. 2.1 is called locally asymptotically stable if it is stable and if there exist $\gamma>0$ such that if

$$
x_{-k}, \cdots x_{-1}, x_{0} \in I
$$

and

$$
\left|x_{-k}-\bar{x}\right|+\left|x_{-k+1}-\bar{x}\right|+\cdots+\left|x_{0}-\bar{x}\right|<\gamma
$$

then

$$
\lim _{n \rightarrow \infty} x_{n}=\bar{x}
$$

3. The equilibrium point $\bar{x}$ of Eq. 2.1 is called global attractor if for every

$$
x_{-k}, \cdots x_{-1}, x_{0} \in I
$$

we have

$$
\lim _{n \rightarrow \infty} x_{n}=\bar{x}
$$

4. The equilibrium point $\bar{x}$ of Eq. 2.1 is called globally asymptotically stable if it is stable and is a global attractor.
5. The equilibrium point $\bar{x}$ of Eq. 2.1 is called unstable if it is not stable
6. The equilibrium point $\bar{x}$ of Eq. 2.1 is called repeller if there exists $r>0$ such that if $x_{-k}, \cdots x_{-1}, x_{0} \in I$ and

$$
\left|x_{-k}-\bar{x}\right|+\left|x_{-k+1}-\bar{x}\right|+\cdots+\left|x_{0}-\bar{x}\right|<\gamma
$$

then there exists $N>-k$ such that

$$
\left|x_{N}-\bar{x}\right|>r
$$

Clearly, a repller is an unstable equilibrium.
Definition 2.4. [16](Linearization)
Let $a=\frac{\partial f}{\partial x}(\bar{x}, \bar{x})$ and $b=\frac{\partial f}{\partial y}(\bar{x}, \bar{x})$ where $f(x, y)$ is the function in Eq.( 2.1) and $\bar{x}$ is the equilibrium of Eq.( 2.1). Then the equation

$$
\begin{equation*}
z_{n+1}=a z_{n}+b z_{n-k}, n=0,1, \cdots \tag{2.2}
\end{equation*}
$$

is called linearized equation associated with Eq.(2.1) about the equilibrium point $\bar{x}$, and its characteristic equation is

$$
\begin{equation*}
\lambda^{k+1}-a \lambda^{k}-b=0 \tag{2.3}
\end{equation*}
$$

### 2.3 Theorems

Theorem 2.2. [16] (Linearized Stability)

1. If all the roots of Eq.(2.3) lie in open disk $|\lambda|<1$, then the equilibrium point $\bar{x}$ of Eq.(2.1) is asymptotically stable.
2. If at least one root of Eq.(2.3) has absolute value greater than 1, then the equilibrium $\bar{x}$ of Eq.(2.1) is unstable

Theorem 2.3. [3] Assume $a, b \in R$ and $k \in\{1,2, \cdots\}$. Then

$$
\begin{equation*}
|a|+|b|<1 \tag{2.4}
\end{equation*}
$$

is sufficient condition for asymptotic stability of the difference equation

$$
\begin{equation*}
x_{n+1}-a x_{n}+b x_{n-k}=0, n=0,1,2, \cdots \tag{2.5}
\end{equation*}
$$

suppose in addition that one of the following two cases holds:

1. $k$ is odd and $b<0$.
2. $k$ is even and $a b<0$.

Then 2.4 is a necessary condition for asymptotic stability of Eq.(2.5)
Theorem 2.4. [15] The difference equation

$$
y_{n+1}-b y_{n}+b y_{n-k}=0, n=0,1,2, \ldots
$$

is asymptotically stable iff $0<|b|<\frac{1}{2} \cos \left(\frac{\pi}{k+2}\right)$
Theorem 2.5. [11] consider the difference equation (2.1). Let $I=[a, b]$ be some interval of real numbers and assume that

$$
f:[a, b] \times[a, b] \rightarrow[a, b]
$$

is continuous function satisfying the following properties:

1. $f(x, y)$ is non increasing in $x$ for each $y \in[a, b]$ and $f(x, y)$ is non increasing in $y$ for each $x \in[a, b]$
2. If $(m, M) \in[a, b] \times[a, b]$ is a solution of the system

$$
\begin{aligned}
m & =f(M, M) \\
M & =f(m, m)
\end{aligned}
$$

then $m=M$.
3. The equation $f(x, x)=x$ has a unique positive solution.

Then Eq.( 2.1) has a unique positive solution and every positive solution of Eq.( 2.1) converges to $\bar{x}$.

Proof. set $m_{0}=a$ and $M_{0}=b$. for $i=1,2,3, \cdots$

$$
m_{i}=f\left(M_{i-1}, M_{i-1}\right) \text { and } M_{i}=f\left(m_{i-1}, m_{i-1}\right)
$$

Then

$$
m_{1}=f\left(M_{0}, M_{0}\right) \geq a=m_{0} \text { and } M_{1}=f\left(m_{0}, m_{0}\right) \leq b=M_{0}
$$

and

$$
\begin{aligned}
& m_{2}=f\left(M_{1}, M_{1}\right) \geq f\left(M_{0}, M_{0}\right)=m_{1} \geq m_{0} \\
& M_{2}=f\left(m_{1}, m_{1}\right) \leq f\left(m_{0}, m_{0}\right)=M_{1} \leq M_{0}
\end{aligned}
$$

By induction, we have

$$
m_{0} \leq m_{1} \cdots m_{i} \leq \cdots \leq M_{i} \leq \cdots \leq M_{1} \leq M_{0}
$$

Also

$$
\begin{aligned}
& x_{n+1}=f\left(x_{n}, x_{n-k}\right) \leq f\left(m_{0}, m_{0}\right)=M_{1} \\
& x_{n+1}=f\left(x_{n}, x_{n-k}\right) \geq f\left(M_{0}, M_{0}\right)=m_{1}
\end{aligned}
$$

and

$$
\begin{aligned}
& x_{n+1}=f\left(x_{n}, x_{n-k}\right) \leq f\left(m_{1}, m_{1}\right)=M_{2} \\
& x_{n+1}=f\left(x_{n}, x_{n-k}\right) \geq f\left(M_{1}, M_{1}\right)=m_{2}
\end{aligned}
$$

By induction, we have

$$
m_{i} \leq x_{n} \leq M_{i}, \quad n \geq(i-1) k+i
$$

set

$$
m=\lim _{i \rightarrow \infty} m_{i} \text { and } M=\lim _{i \rightarrow \infty} M_{i}
$$

then we have

$$
m \leq \lim _{i \rightarrow \infty} \inf x_{i} \leq \lim _{i \rightarrow \infty} \sup x_{i} \leq M
$$

By continuity of $f$

$$
m=f(M, M) \text { and } M=f(m, m)
$$

by assumption (2)

$$
m=M=\bar{x}
$$

Theorem 2.6. [11] Let I be an interval of real numbers and assume

$$
f:[a, b] \times[a, b] \rightarrow[a, b]
$$

is continuously function satisfying the following properties

1. $f(x, y)$ is non decreasing in $x$ for each $y \in[a, b]$ and $f(x, y)$ is non increasing in $y$ for each $x \in[a, b]$
2. If $(m, M) \in[a, b] \times[a, b]$ is a solution of the system

$$
\begin{aligned}
m & =f(m, M) \\
M & =f(M, m)
\end{aligned}
$$

then $m=M$.
Then Eq.(2.1) has a unique equilibrium $\bar{x} \in[a, b]$ and every solution of Eq.(2.1) converges to $\bar{x}$

Proof. set

$$
m_{0}=a \text { and } M_{0}=b
$$

for each $i=1,2,3, \cdots$

$$
m_{i}=f\left(m_{i-1}, M_{i-1}\right) \text { and } M_{i}=f\left(M_{i-1}, m_{i-1}\right)
$$

Then

$$
m_{1}=f\left(m_{0}, M_{0}\right) \geq a=m_{0} \text { and } M_{1}=f\left(M_{0}, m_{0}\right) \leq b=M_{0}
$$

and

$$
\begin{aligned}
& m_{2}=f\left(m_{1}, M_{1}\right) \geq f\left(m_{0}, M_{0}\right)=m_{1} \geq m_{0} \\
& M_{2}=f\left(M_{1}, m_{1}\right) \leq f\left(M_{0}, m_{0}\right)=M_{1} \leq M_{0}
\end{aligned}
$$

By induction, we have

$$
m_{0} \leq m_{1} \cdots m_{i} \leq \cdots \leq M_{i} \leq \cdots \leq M_{1} \leq M_{0}
$$

Also

$$
\begin{aligned}
& x_{n+1}=f\left(x_{n}, x_{n-k}\right) \leq f\left(M_{0}, m_{0}\right)=M_{1} \\
& x_{n+1}=f\left(x_{n}, x_{n-k}\right) \geq f\left(m_{0}, M_{0}\right)=m_{1}
\end{aligned}
$$

and

$$
\begin{aligned}
x_{n+1} & =f\left(x_{n}, x_{n-k}\right) \leq f\left(M_{1}, m_{1}\right)=M_{2} \\
x_{n+1} & =f\left(x_{n}, x_{n-k}\right) \geq f\left(m_{1}, M_{1}\right)=m_{2}
\end{aligned}
$$

By induction, we have

$$
m_{i} \leq x_{n} \leq M_{i}, \quad n \geq(i-1) k+i
$$

set

$$
m=\lim _{i \rightarrow \infty} m_{i} \text { and } M=\lim _{i \rightarrow \infty} M_{i}
$$

then we have

$$
m \leq \lim _{i \rightarrow \infty} \inf x_{i} \leq \lim _{i \rightarrow \infty} \sup x_{i} \leq M
$$

By continuity of $f$

$$
m=f(m, M) \text { and } M=f(M, m)
$$

therefore in view of (2)

$$
m=M=\bar{x}
$$

Theorem 2.7. [5] Consider

$$
y_{n+1}=f\left(y_{n}, y_{n-k}\right), n=0,1,2, \ldots
$$

where $k \in\{1,2, \ldots\}$. Let $I=[a, b]$ be some interval of positive real numbers and assume that

$$
f:[a, b] \times[a, b] \rightarrow[a, b]
$$

is continuous function satisfying the following properties:

1. $f(u, v)$ is nonincreasing in $u$ and nondecreasing in $v$.
2. If $(m, M) \in[a, b]$ is a solution of the system

$$
m=f(M, m) \text { and } M=f(m, M)
$$

Then

$$
m=M
$$

Then the equation $y_{n+1}=f\left(y_{n}, y_{n-k}\right)$ has a unique positive equilibrium $\bar{y}$ and every solution converges to $\bar{y}$.

Theorem 2.8. Assume that $f \in C[(0, \infty) \times(0, \infty),(0, \infty)]$ is such that : $f(x, y)$ is decreasing in $x$ for each fixed $y$.and $f(x, y)$ is increasing in $y$ for each fixed $x$. Let $\bar{x}$ be a positive equilibrium of equation (2.1) then except possibly for the first semicycle, every oscillatory solution of equation (2.1) has semicycle of length $k$

Proof. When $k=1$, the proof is presented as Theorem1.7.1 in [11]. When $k=2$, the proof is presented as Theorem4 in [1]. We just give the proof of the theorem for $k=3$. The other cases for $k \geq 4$ are similar and can be omitted.

Let $\left\{x_{n}\right\}$ be a solution of equation( 2.1) with at least four semicycles. Then there exists $N>0$ such that either

$$
x_{N-1}<\bar{x} \leq x_{N+2}
$$

or

$$
x_{N-1} \geq \bar{x}>x_{N+2}
$$

we will assume that :

$$
x_{N-1}<\bar{x} \leq x_{N+2}
$$

other cases is similar and will be omitted. The by using monotonic character of $f(x, y)$ we have

$$
x_{N+3}=f\left(x_{N+2}, x_{N-1}\right)<f(\bar{x}, \bar{x})=\bar{x}
$$

and

$$
x_{N+4}=f\left(x_{N+3}, x_{N}\right)>f(\bar{x}, \bar{x})=\bar{x}
$$

thus

$$
x_{N+3}<\bar{x}<x_{N+4}
$$

The proof is complete.
Theorem 2.9. Assume that $f \in C[(0, \infty) \times(0, \infty),(0, \infty)]$ and that : $f(x, y)$ is decreasing in both arguments. Let $\bar{x}$ be a positive equilibrium of equation (2.1) then every oscillatory solution of equation (2.1) has semicycle of length at most $k$.

Proof. When $k=1$, the proof is presented as Theorem1.7.2 in [11]. We just give the proof of the theorem for $k=2$. The other cases for $k \geq 3$ are similar and can be omitted. Assume that $\left\{x_{n}\right\}$ is an oscillatory solution with three consecutive terms $x_{N-1}, x_{N-1}, x_{N+1}$ in a positive semicycle

$$
x_{N-2} \geq \bar{x}, x_{N-1} \geq \bar{x}, x_{N} \geq \bar{x}
$$

with at least one of the inequalities being strict. The proof in the case of negative semicycle is similar and is omitted.

Then by using the decreasing character of $f$. We obtain

$$
x_{N+2}=f\left(x_{N+1}, x_{N-1}\right)<f(\bar{x}, \bar{x})=\bar{x}
$$

which completes the proof.

For $k=3$ assume that $\left\{x_{n}\right\}$ is an oscillatory solution with four consecutive terms $x_{N-1}, x_{N}, x_{N+1}, x_{N+2}$ in a negative semicycle

$$
x_{N-1} \leq \bar{x}, x_{N} \leq \bar{x}, x_{N+1} \leq \bar{x}, x_{N+2} \leq \bar{x}
$$

with at least one of the inequalities being strict. The proof in the case of positive semicycle is similar and is omitted. Then by using the decreasing character of $f$. We obtain

$$
x_{N+3}=f\left(x_{N+2}, x_{N-1}\right)>f(\bar{x}, \bar{x})=\bar{x}
$$

which completes the proof.

Theorem 2.10. Assume that $f \in C[(0, \infty) \times(0, \infty),(0, \infty)]$ is such that : $f(x, y)$ is increasing in $x$ for each fixed $y$.and $f(x, y)$ is decreasing in $y$ for each fixed $x$. Let $\bar{x}$ be a positive equilibrium of equation ( 3.3) then every oscillatory solution of equation (3.3) has semicycle of length at least $k$

Proof. When $k=1$, the proof is presented as Theorem1.7.4 in [11]. We just give the proof of the theorem for $k=2$.the other cases for $k \geq 3$ are similar and can be omitted.

Assume that $\left\{x_{n}\right\}$ is an oscillatory solution with three consecutive terms

$$
x_{N-1}, x_{N}, x_{N+1}
$$

such that

$$
x_{N-1}<\bar{x}<x_{N+1}
$$

or

$$
x_{N-1}>\bar{x}>x_{N+1}
$$

we will assume that

$$
x_{N-1}<\bar{x}<x_{N+1}
$$

the other case is similar and will be omitted. Then by using decreasing character of $f$ we obtain

$$
x_{N+2}=f\left(x_{N+1}, x_{N-1}\right)>f(\bar{x}, \bar{x})
$$

Now, if $x_{N} \geq \bar{x}$ then the result follows. Otherwise $x_{N}<\bar{x}$. Hence

$$
x_{N+3}=f\left(x_{N+2}, x_{N}\right)>f(\bar{x}, \bar{x})=\bar{x}
$$

which shows that it has at least three terms in the positive semicycle

## Chapter 3

## Dynamics of $x_{n+1}=\frac{\alpha+\beta x_{n}+\gamma x_{n-k}}{B x_{n}+C x_{n-k}}$

In this chapter and chapter 5 we present the main part of this theses, that is studying and investigating the difference equation

$$
\begin{equation*}
x_{n+1}=\frac{\alpha+\beta x_{n}+\gamma x_{n-k}}{B x_{n}+C x_{n-k}}, n=0,1,2, \cdots \tag{3.1}
\end{equation*}
$$

where where the parameters $\alpha, \beta, \gamma, B, C$, and the initial conditions $x_{-k}, x_{-k+1}, \cdots, x_{0}$ are nonnegative real numbers, $\mathrm{k}=\{1,2,3 \cdots\}$.

This chapter includes mathematical issues, and methodologies that used in such monographs.

### 3.1 Change of variables

The change of variable

$$
x_{n}=\frac{\beta}{B} y_{n}
$$

reduces Eq.(3.1) to the difference equation

$$
\begin{equation*}
y_{n+1}=\frac{p+y_{n}+l y_{n-k}}{y_{n}+q y_{n-k}}, n=0,1,2,3, \ldots \tag{3.2}
\end{equation*}
$$

where

$$
p=\frac{\alpha B}{\beta^{2}}, q=\frac{C}{B}, l=\frac{\gamma}{\beta}
$$

with $p, q \in(0, \infty)$ and

$$
y_{k}, y_{-k+1}, \cdots, y_{-1}, y_{0} \in(0, \infty)
$$

Proof. Since

$$
\begin{aligned}
x_{n} & =\frac{\beta}{B} y_{n} \\
x_{n+1} & =\frac{\beta}{B} y_{n+1} \\
x_{n-k} & =\frac{\beta}{B} y_{n-k}
\end{aligned}
$$

substitute in the Eq.(3.1). We get

$$
\frac{\beta}{B} y_{n+1}=\frac{\alpha+\beta \frac{\beta}{B} y_{n}+\gamma \frac{\beta}{B} y_{n-k}}{B \frac{\beta}{B} y_{n}+C \frac{\beta}{B} y_{n-k}}
$$

by pulling a common factor $\frac{\beta}{B}$,

$$
\frac{\beta}{B} y_{n+1}=\frac{\frac{\beta}{B}\left(\frac{B}{\beta} \alpha+\beta y_{n}+\gamma y_{n-k}\right)}{\frac{\beta}{B}\left(B y_{n}+C y_{n-k}\right)}
$$

hence

$$
\frac{\beta}{B} y_{n+1}=\frac{\beta\left(\frac{B}{\beta^{2}} \alpha+y_{n}+\frac{\gamma}{\beta} y_{n-k}\right)}{B\left(y_{n}+\frac{C}{B} y_{n-k}\right)}
$$

Let

$$
p=\frac{\alpha B}{\beta^{2}}, q=\frac{C}{B}, l=\frac{\gamma}{\beta}
$$

reduces the Eq.(3.1) to

$$
y_{n+1}=\frac{p+y_{n}+l y_{n-k}}{y_{n}+q y_{n-k}}, n=0,1,2, \cdots
$$

The proof has been completed.

### 3.2 Equilibrium Points

In this section we investigate the equilibrium point of the nonlinear difference equation

$$
\begin{equation*}
y_{n+1}=\frac{p+y_{n}+l y_{n-k}}{y_{n}+q y_{n-k}}, n=0,1,2, \cdots \tag{3.3}
\end{equation*}
$$

where the parameters $p, q, l$ and the initial conditions $y_{-k}, y_{-k+1}, \cdots, y_{-1}, y_{0}$ are nonnegative real numbers, $k=\{1,2,3, \cdots\}$. To find the equilibrium point in view of its definition, we solve the following equation

$$
\bar{y}=\frac{p+\bar{y}+l \bar{y}}{\bar{y}+q \bar{y}}
$$

by cross multiplication, we get

$$
\begin{equation*}
\bar{y}^{2}(1+q)=p+\bar{y}(1+l) \tag{3.4}
\end{equation*}
$$

by rearranging the terms, we get

$$
(1+q) \bar{y}^{2}-(1+l) \bar{y}-p=0
$$

now, we use the quadratic formula to solve the above equation

$$
\bar{y}=\frac{(1+l) \pm \sqrt{(1+l)^{2}+4 p(1+q)}}{2(1+q)}
$$

Hence, the only positive equilibrium point of Eq.( 3.3) is:

$$
\begin{equation*}
\bar{y}=\frac{(1+l)+\sqrt{(1+l)^{2}+4 p(1+q)}}{2(1+q)} \tag{3.5}
\end{equation*}
$$

### 3.3 Linearization

For our investigation. Let

$$
f(u, v)=\frac{p+u+l v}{u+q v}
$$

is the function in Eq.( 3.3)
Definition 3.1. Linearized Equation The equation

$$
\begin{equation*}
z_{n+1}=\frac{\partial f}{\partial u}(\bar{y}, \bar{y}) z_{n}+\frac{\partial f}{\partial v}(\bar{y}, \bar{y}) z_{n-k} \tag{3.6}
\end{equation*}
$$

is called the linearized equation associated with Eq.( 3.3) about the equilibrium point $\bar{x}$
since

$$
f(u, v)=\frac{p+u+l v}{u+q v}
$$

we have,

$$
\begin{aligned}
\frac{\partial f}{\partial u} & =\frac{(u+q v)-(p+u+l v)}{(u+q v)^{2}} \\
& =\frac{u+q v-p-u-l v}{(u+q v)^{2}} \\
& =\frac{q v-l v-p}{(u+q v)^{2}} \\
& =\frac{(q-l) v-p}{(u+q v)^{2}}
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial f}{\partial v} & =\frac{l(u+q v)-q(p+u+l v)}{(u+q v)^{2}} \\
& =\frac{l u+l q v-p q-u q-l q v}{(u+q v)^{2}} \\
& =\frac{l u-q u-p q}{(u+q v)^{2}} \\
& =\frac{(l-q) u-p q}{(u+q v)^{2}} \\
& =-\frac{(q-l) u+p q}{(u+q v)^{2}}
\end{aligned}
$$

hence

$$
\begin{aligned}
\frac{\partial f}{\partial u}(\bar{y}, \bar{y}) & =\frac{(q-l) \bar{y}-p}{(\bar{y}+q \bar{y})^{2}} \\
& =\frac{(q-l) \bar{y}-p}{\bar{y}^{2}(1+q)^{2}}
\end{aligned}
$$

but, by rearranging Eq.( 3.4), we get

$$
\begin{equation*}
\bar{y}^{2}=\frac{p+\bar{y}(1+l)}{1+q} \tag{3.7}
\end{equation*}
$$

then

$$
\begin{aligned}
\frac{\partial f}{\partial u}(\bar{y}, \bar{y}) & =\frac{(q-l) \bar{y}-p}{(1+q)^{2}\left(\frac{p+\bar{y}(1+l)}{1+q}\right)} \\
& =\frac{(q-l) \bar{y}-p}{(1+q)(p+(1+l) \bar{y})}
\end{aligned}
$$

now

$$
\begin{aligned}
\frac{\partial f}{\partial v}(\bar{y}, \bar{y}) & =-\frac{(q-l) \bar{y}+p q}{(\bar{y}+q \bar{y})^{2}} \\
& =-\frac{(q-l) \bar{y}+p q}{\bar{y}^{2}(1+q)^{2}} \\
& =-\frac{(q-l) \bar{y}+p q}{(1+q)^{2}\left(\frac{p+\bar{y}(1+l)}{1+q}\right)} \\
& =-\frac{(q-l) \bar{y}+p q}{(1+q)(p+(1+l) \bar{y})}
\end{aligned}
$$

The Linearized Equation associated with equation 3.3 about the equilibrium point $\bar{x}$ is:

$$
z_{n+1}=\frac{(q-l) \bar{y}-p}{(1+q)(p+(1+l) \bar{y})} z_{n}+-\frac{(q-l) \bar{y}+p q}{(1+q)(p+(1+l) \bar{y})} z_{n-k}
$$

i.e

$$
\begin{equation*}
z_{n+1}-\frac{(q-l) \bar{y}-p}{(1+q)(p+(1+l) \bar{y})} z_{n}+\frac{(q-l) \bar{y}+p q}{(1+q)(p+(1+l) \bar{y})} z_{n-k}=0 \tag{3.8}
\end{equation*}
$$

and its characteristic equation is :

$$
\begin{equation*}
\lambda^{n+1}-\frac{(q-l) \bar{y}-p}{(1+q)(p+(1+l) \bar{y})} \lambda^{n}+\frac{(q-l) \bar{y}+p q}{(1+q)(p+(1+l) \bar{y})} \lambda^{n-k}=0 \tag{3.9}
\end{equation*}
$$

### 3.4 Local Stability

The aim of this section is to establish the stability of positive equilibrium point of eq.( 3.3). In sections 3.2 and 3.3, We have fined that Eq.( 3.3) has the only positive equilibrium point:

$$
\bar{y}=\frac{(1+l)+\sqrt{(1+l)^{2}+4 p(1+q)}}{2(1+q)}
$$

and the linearized equation is given by:

$$
z_{n+1}-\frac{(q-l) \bar{y}-p}{(1+q)(p+(1+l) \bar{y})} z_{n}+\frac{(q-l) \bar{y}+p q}{(1+q)(p+(1+l) \bar{y})} z_{n-k}=0
$$

where

$$
\frac{(q-l) \bar{y}-p}{(1+q)(p+(1+l) \bar{y})}=a, \frac{(q-l) \bar{y}+p q}{(1+q)(p+(1+l) \bar{y})}=b
$$

it is necessary to mention that the equilibrium point of Eq.(3.3) is locally asymptotically stable for all values of the parameters $p$ and $q$ when $\mathbf{k}=1$.(see [6])

The following theorem is a direct consequence of theorems (2.2) and (2.3).
Theorem 3.1. The unique equilibrium point $\bar{y}$ of eq. (3.3) is locally asymptotically stable in the following cases :

1. $q>l$ there are two cases

- $l>1$ implies $q>1$
- $q<1 \& l<1$.

2. $q<l$

- if $\bar{y}(l-q)<p q$ i.e. $\bar{y}<\frac{p q}{l-q}$
- if $\bar{y}(l-q)>p q$ and $\bar{y}<\frac{2 p q}{l-3 q-1-q l}$ i.e. $\bar{y}>\frac{p q}{l-q}$ and $\bar{y}<\frac{2 p q}{l-3 q-1-q l}$

It is important to note that

- When $k$ is odd and $q<l$. Then $\bar{y}>\frac{p q}{l-q}$ is necessary and condition for local stability.
- When $k$ is even and $q>l$. Then $\bar{y}<\frac{p}{q-l}$ is necessary and condition for local stability.
- When $k$ is even and $q<l$. Then $\bar{y}<\frac{p q}{l-q}$ is necessary and condition for local stability.

Proof. By theorem (2.3)

1. When $q>l$ there are two cases:

- $(q-l) \bar{y}>p$ i.e. $\bar{y}>\frac{p}{q-l}$

$$
|a|+|b|<1
$$

by substituting values of $a$ and $b$, we have

$$
\frac{(q-l) \bar{y}-p}{(1+q)(p+(1+l) \bar{x})}+\frac{(q-l) \bar{y}+p q}{(1+q)(p+(1+l) \bar{y})}<1
$$

then

$$
\frac{(q-l) \bar{y}-p+(q-l) \bar{y}+p q}{(1+q)(p+(1+l) \bar{y})}<1
$$

by multiplying, we have

$$
(q-l) \bar{y}-p+(q-l) \bar{y}+p q<(1+q)(p+(1+l) \bar{y})
$$

then

$$
2(q-l) \bar{y}-p+p q<p+(1+l) \bar{y}+p q+q(1+l) \bar{y}
$$

so

$$
2(q-l) \bar{y}-2 p<\bar{y}(1+l)(1+q)
$$

hence

$$
-2 p<\bar{y}[(1+l)(1+q)-2(q-l)]
$$

then

$$
-2 p<\bar{y}[1+l+q+l q-2 q+2 l]
$$

so

$$
-2 p<\bar{y}[1-q+3 l+l q]
$$

note that when $l>1$, then $\bar{y}[1-q+3 l+l q]$ is strictly greater than zero. And when $q<1$, also implies $\bar{y}[1-q+3 l+l q]$ is strictly greater than zero

- $(q-l) \bar{y}<p$, i.e $\bar{y}<\frac{p}{q-l}$ We have the following inequality :

$$
\frac{p-(q-l) \bar{y}}{(1+q)(p+(1+l) \bar{y})}+\frac{(q-l) \bar{y}+p q}{(1+q)(p+(1+l) \bar{y})}<1
$$

then

$$
\frac{p-(q-l) \bar{y}+(q-l) \bar{y}+p q}{(1+q)(p+(1+l) \bar{y})}<1
$$

hence

$$
p+p q<p+(1+l) \bar{y}+p q+q(1+l) \bar{y}
$$

cancel the common terms in both sides, we get

$$
0<+(1+l) \bar{y}+q(1+l) \bar{y}
$$

which is true for all values of $l, q, \bar{x}$
2. When $q<l$
there are two cases:

- when $\bar{y}<\frac{p q}{l-q}$

$$
\begin{aligned}
& |a|=\frac{(l-q) \bar{y}+p}{(1+q)[p+(1+l) \bar{y}]} \text { and }|b|=\frac{p q-(l-q) \bar{y}}{(1+q)[p+(1+l) \bar{y}]} \\
& \Longrightarrow \\
& \frac{(l-q) \bar{y}+p}{(1+q)[p+(1+l) \bar{y}]}+\frac{p q-(l-q) \bar{y}}{(1+q)[p+(1+l) \bar{y}]}<1 \\
& \Rightarrow \\
& \frac{(l-q) \bar{y}+p+p q-(l-q) \bar{y}}{(1+q)[p+(1+l) \bar{y}]}<1 \\
& \Rightarrow \\
& \frac{p+p q}{(1+q)[p+(1+l) \bar{y}]}<1 \\
& \Rightarrow \\
& p+p q<(1+q)[p+(1+l) \bar{y}] \\
& \Rightarrow \\
& p(1+q)<(1+q)[p+(1+l) \bar{y}] \\
& \Rightarrow \\
& p<[p+(1+l) \bar{y}] \\
& \Longrightarrow \\
& 0<(1+l) \bar{y}
\end{aligned}
$$

- when $\bar{y}>\frac{p q}{l-q}$

$$
\left.\begin{array}{ll}
|a|= & \frac{(l-q) \bar{y}+p}{(1+q)[p+(1+l) \bar{y}]} \text { and }|b|=\frac{(l-q) \bar{y}-p q}{(1+q)[p+(1+l \bar{y}]} \\
& \begin{array}{c}
(l-q) \bar{y}+p \\
\hline 1+q)[p+(1+l) \bar{y}]
\end{array}+\frac{(l-q) \bar{y}-p q}{(1+q)[p+(1+l) \bar{y}]}<1 \\
\Longrightarrow & \frac{(l-q) \bar{y}+p+(l-q) \bar{y}-p q}{(1+q)[p+(1+l) \bar{y}]}<1 \\
\Longrightarrow & 2(l-q) \bar{y}+p-p q<(1+q)[p+(1+l) \bar{y}]
\end{array}\right] \begin{gathered}
2(l-q) \bar{y}+p-p q<p+(1+l) \bar{y}+p q+q(1+l) \bar{y} \\
\Longrightarrow
\end{gathered} \begin{gathered}
2(l-q) \bar{y}-(1+l) \bar{y}-q(1+l) \bar{y}<2 p q \\
\Longrightarrow
\end{gathered}
$$

$$
\bar{y}<\frac{2 p q}{l-3 q-1-q l}
$$

We have investigated the two cases $q>l$ and $q<l$ in previous theorem. The next theorem about case $q=l$. When $q=l$ the Eq.(3.3) becomes

$$
\begin{equation*}
y_{n+1}=\frac{p+y_{n}+q y_{n-k}}{y_{n}+q y_{n-k}} \tag{3.10}
\end{equation*}
$$

and the positive equilibrium point $\bar{y}=\frac{\left.1+q+\sqrt{( }(1+q)^{2}+4 p(1+q)\right)}{2(1+q)}$. Observe that $\bar{y}>1$.

Theorem 3.2. Assume that $q=l$

1. Suppose that $k$ is odd. Then the equilibrium point $\bar{y}$ of Eq.(3.10) is asymptotically stable.
2. Suppose that $k$ is even. Then the equilibrium point $\bar{y}$ of Eq.(3.10) is asymptotically stable iff $q=l$.
Proof. Let $f(x, y)=\frac{p+x+q y}{x+q y}$. Assume $a=\frac{\partial f}{\partial x}(\bar{y}, \bar{y})=\frac{-p}{(1+q)[p+(1+q) \bar{y}]}$ and $b=\frac{\partial f}{\partial y}(\bar{y}, \bar{y})=$ $\frac{-p q}{(1+q)[p+(1+q) \overline{\bar{y}}}$. Observe that $a<0$ and $a b>0$. Then the proof is a direct sequence of theorem 2.3. the proof is complete.

### 3.5 Invariant Intervals

The fundamental idea if invariant interval is widely understood, and for the sake of clarity, we give the following definition which will be the key concepts in this monograph.

Definition 3.2. [3](Invariant interval) An invariant interval for the difference equation (2.1) is an interval $I$ with the property that if two consecutive terms of the solution fall in $I$ then all subsequence terms of the solution also belong to $I$. In other words, $I$ is an invariant interval for Eq.( 2.1) if $x_{N-k+1}, \cdots, x_{N-1}, x_{N} \in I$ for some $N \geq 0$, then $x_{n} \in I$ for every $n>N$.
Theorem 3.3. Let $\left\{y_{n}\right\}_{n=-k}^{\infty}$ be a solution of Eq.(3.3). Then the following statements are true:

1. Suppose $p+l<q$ and assume that for some $N>0$,

$$
y_{N-k}, y_{N-k+1}, \cdots, y_{N} \in\left[\frac{p+l}{q}, 1\right]
$$

$$
\text { then } y_{n} \in\left[\frac{p+l}{q}, 1\right] \text { for all } n>N
$$

2. Suppose $p+l>q$ and assume that for some $N>0$,

$$
y_{N-k}, y_{N-k+1}, \cdots, y_{N} \in\left[1, \frac{p+l}{q}\right]
$$

then $y_{n} \in\left[1, \frac{p+l}{q}\right]$ for all $n>N$

Proof. 1. The basic ingredient behind the proof is the fact that when $u, v \in$ $\left[\frac{p+l}{q}, \infty\right)$, the function

$$
f(u, v)=\frac{u+p+l v}{u+q v}
$$

is increasing in $u$ and decreasing in $v$. If for some $N>0, \frac{p+l}{q} \leq y_{N-k}, y_{N-k+1}, \cdots, y_{N} \leq$ 1 then

$$
\begin{aligned}
y_{N+1} & =\frac{p+y_{N}+l y_{N-k}}{y_{N}+q y_{N-k}} \\
& \leq \frac{p+y_{N}+l}{y_{N}+q \frac{p+l}{q}} \\
& =1
\end{aligned}
$$

and

$$
\begin{aligned}
y_{N+1} & =\frac{p+y_{N}+l y_{N-k}}{y_{N}+q y_{N-k}} \\
& \geq \frac{\frac{p+l}{q}+p+l}{\frac{p+l}{q}+q} \\
& =\frac{(p+l)\left[\frac{1}{q}+1\right]}{q\left[1+\frac{1}{q} \frac{p+l}{q}\right]} \\
& >\frac{p+l}{q}
\end{aligned}
$$

The proof is follow by induction.
2. If for some $N>0,1 \leq y_{N-k}, y_{N-k+1}, \cdots, y_{N} \leq \frac{p+l}{q}$ then

$$
\begin{aligned}
y_{N+1} & =\frac{p+y_{N}+l y_{N-k}}{y_{N}+q y_{N-k}} \\
& \leq \frac{\frac{p+l}{q}+p+l}{1+q} \\
& =\frac{(p+l)\left[\frac{1}{q}+1\right]}{q\left[\frac{1}{q}+1\right]} \\
& =\frac{p+l}{q}
\end{aligned}
$$

and

$$
\begin{aligned}
y_{N+1} & =\frac{p+y_{N}+l y_{N-k}}{y_{N}+q y_{N-k}} \\
& \geq \frac{y_{N}+p+l}{y_{N}+q\left[\frac{p+l}{q}\right]} \\
& =1
\end{aligned}
$$

and the proof follows by induction.
The proof is complete.

### 3.6 Existence of two cycles

In this section we give the necessary and sufficient conditions for Eq.( 3.3) to have a prime period-two solution and we exhibit all prime period-two solutions of the Eq.( 3.3).

Definition 3.3. Let $\left\{x_{n}\right\}_{n=-k}^{\infty}$ be a solution of Eq.( 3.3). We say the solution has a prime period two if the solution eventually take the form:

$$
\cdots, \phi, \psi, \phi, \psi, \phi, \psi, \cdots
$$

where $\phi, \psi$ are distinct and positive.
Theorem 3.4. Two cycles theorem

1. The Eq.( 3.3) has no nonnegative prime period two if
(a) $k$ is even.
(b) $k$ is odd and $l \leq 1$.
(c) $k$ is odd, and $q \geq 1$.
2. If $k$ is odd and $l>1$ and $q<1$, then the Eq.(3.3) has prime period two solution $\cdots, \phi, \psi, \phi, \psi, \phi, \psi, \cdots$ where the values of $\phi$ and $\psi$ (positive and distinct) are solutions of quadratic equation :

$$
r^{2}-\frac{l-1}{q} r+\frac{p q+l-1}{q(1-q)}=0
$$

provided the solution exists.
Proof. 1. Assume for the sake of contradiction that there exist distinctive and positive real numbers $\phi$ and $\psi$ such that

$$
\cdots, \phi, \psi, \phi, \psi, \phi, \psi, \cdots
$$

is a prime period two solution of Eq.(3.3).there are two cases to be considered
(a) $k$ is even in this case $\phi$ and $\psi$ satisfy

$$
\phi=\frac{p+\psi+l \psi}{\psi+q \psi}
$$

and

$$
\psi=\frac{p+\phi+l \phi}{\phi+q \phi}
$$

so

$$
\begin{align*}
& \phi \psi(1+q)=p+\phi+l \phi  \tag{3.11}\\
& \phi \psi(1+q)=p+\psi+l \psi \tag{3.12}
\end{align*}
$$

by subtracting Eq.( 3.12)from Eq.( 3.11), we have

$$
\begin{aligned}
& 0=p+\phi+l \phi-p-\psi-l \psi \\
& 0=\phi-\psi+l(\phi-\psi) \\
& 0=(\phi-\psi)(1+l)
\end{aligned}
$$

that implies $\phi=\psi$ which contradicts the assumption $\phi \neq \psi$
(b) $k$ is odd in this case $\phi$ and $\psi$ satisfy

$$
\phi=\frac{p+\psi+l \phi}{\psi+q \phi}
$$

and

$$
\psi=\frac{p+\phi+l \psi}{\phi+q \psi}
$$

by multiplying, we get

$$
\begin{align*}
& \phi \psi+q \phi^{2}=p+\psi+l \phi  \tag{3.13}\\
& \phi \psi+q \psi^{2}=p+\phi+l \psi \tag{3.14}
\end{align*}
$$

by subtracting Eq.( 3.14) from Eq.( 3.13), we have

$$
\begin{aligned}
q \phi^{2}-q \psi^{2} & =\psi+l \phi-\phi-l \psi \\
q(\phi-\psi)(\phi+\psi) & =l \phi-l \psi+\psi-\phi \\
q(\phi-\psi)(\phi+\psi) & =l(\phi-\psi)-(\phi-\psi) \\
q(\phi-\psi)(\phi+\psi) & =(\phi-\psi)(l-1)
\end{aligned}
$$

$\Rightarrow$

$$
\begin{equation*}
\phi+\psi=\frac{l-1}{q} \tag{3.15}
\end{equation*}
$$

observe that if $l \leq 1$ then $\phi+\psi \leq 0$ which contradicts the hypothesis that $\phi, \psi$ are positive and distinctive.and this proves case ( 1 b ).
(c) Adding Eq.( 3.13) and Eq.( 3.14), we have:

$$
\begin{aligned}
2 \phi \psi+q \phi^{2}+q \psi^{2} & =2 p+\phi+\psi+l \phi+l \psi \\
2 \phi \psi+q\left(\phi^{2}+\psi^{2}\right) & =2 p+\phi+\psi+l(\phi+\psi) \\
2 \phi \psi+q\left(\phi^{2}+\psi^{2}+2 \phi \psi-2 \phi \psi\right) & =2 p+\phi+\psi+l(\phi+\psi) \\
2 \phi \psi-2 q \phi \psi+q\left(\phi^{2}+\psi^{2}+2 \phi \psi\right) & =2 p+(1+l)(\phi+\psi) \\
\phi \psi(2-2 q)+q(\phi+\psi)^{2} & =2 p+(1+l)\left(\frac{l-1}{q}\right) \\
\phi \psi(2-2 q)+q\left(\frac{l-1}{q}\right)^{2} & =2 p+(1+l)\left(\frac{l-1}{q}\right) \\
\phi \psi(2-2 q) & =2 p+(1+l)\left(\frac{l-1}{q}\right)-q\left(\frac{l-1}{q}\right)^{2} \\
\phi \psi(2-2 q) & =2 p+(1+l)\left(\frac{l-1}{q}\right)-\frac{(l-1)^{2}}{q} \\
\phi \psi(2-2 q) & =2 p+\frac{(l-1)[(l+1)-(l-1)]}{q} \\
\phi \psi(2-2 q) & =2 p+\frac{2(l-1)}{q} \\
\phi \psi(2-2 q) & =\frac{2 p q+2(l-1)}{q} \\
\phi \psi & =\frac{2 p q+2(l-1)}{q(2-2 q)}
\end{aligned}
$$

thus

$$
\begin{equation*}
\phi \psi=\frac{p q+(l-1)}{q(1-q)} \tag{3.16}
\end{equation*}
$$

observe that if $q>1$ then $\psi \phi<0$ which contradicts the hypothesis that $\phi a n d \psi$ are positive and distinctive.
2. If $q<1$ and $l>1$, then we have Eq.( 3.15)and Eq.( 3.16). Now, construct the quadratic equation:

$$
r^{2}-\frac{l-1}{q} r+\frac{p q+(l-1)}{q(1-q)}=0
$$

hence, the values of $\phi$ and $\psi$ are the solutions of the above quadratic equation. i.e.

$$
r=\frac{\frac{l-1}{q} \pm \sqrt{\left(\frac{l-1}{q}\right)^{2}-4 \frac{p q+(l-1)}{q(1-q)}}}{2}
$$

the proof is complete.

### 3.7 Analysis of Semicycle and oscillation

We strongly believe a semicycle analysis of the solutions of a scalar difference equation is a powerful tool for a detailed understanding of the solutions and often leads to straightforward proofs of their long term behavior.

In this section, we investigate semicycles and we present some results about the semicycle character of solutions of the difference equation (2.1) under appropriate hypotheses on the function $f$.

Definition 3.4. [15] Semicycle: Let $\left\{y_{n}\right\}_{n=-k}^{\infty}$ be a solution of equation (2.1) and $\bar{y}$ be a positive equilibrium point. We now give the definitions of positive and negative semicycle of a solution of equation (2.1) relative to the equilibrium point $\bar{y}$

- A positive semicycle of a solution $\left\{y_{n}\right\}_{n=-k}^{\infty}$ of equation (2.1) consists of a "string" of terms $\left\{y_{l}, y_{l+1}, \cdots, y_{m}\right\}$, all greater than or equal to the equilibrium $\bar{y}$, with $l \geq-k$ and $m \leq \infty$ and such that

$$
\text { either } l=-k \text {, or } l>-k \text { and } x_{l-1}<\bar{y}
$$

and

$$
\text { either } m=\infty \text {, or } m<\infty \text { and } y_{m+1}<\bar{y}
$$

- A negative semicycle of a solution $\left\{y_{n}\right\}_{n=-k}^{\infty}$ of equation (2.1) consists of a "string" of terms $\left\{y_{l}, y_{l+1}, \cdots, y_{m}\right\}$, all less than to the equilibrium $\bar{y}$, with $l \geq-k$ and $m \leq \infty$ and such that

$$
\text { either } l=-k \text {, or } l>-k \text { and } x_{l-1} \geq \bar{y}
$$

and

$$
\text { either } m=\infty, \text { or } m<\infty \text { and } y_{m+1} \geq \bar{y}
$$

The first semi-cycle of a solution starts with the term $y_{k}$ and is positive if $y_{k} \geq \bar{y}$ and negative if $y_{-k}<\bar{y}$.

Definition 3.5. [11](Oscillation)

1. A sequence $\left\{x_{n}\right\}$ is said to oscillate about zero or simply to oscillate if the terms $x_{n}$ are neither eventually all positive nor eventually all negative. Otherwise the sequence is called nonoscillatory. A sequence is called strictly oscillatory if for $n_{0}$, there exist $n_{1}, n_{2} \geq n_{0}$ such that $x_{n_{1}} x_{n_{2}}<0$.
2. A sequence $x_{n}$ is said to oscillate about $\bar{x}$ if the sequence $x_{n}-\bar{x}$ oscillate. The sequence $x_{n}$ is called strictly oscillatory about $\bar{x}$ if the sequence $x_{n}-\bar{x}$ is strictly oscillatory.

Again The aim of this section is to present the analysis of semicycles of solution of Eq.(3.3) relative to equilibrium point $\bar{y}$ and based on invariant interval of Eq.( 3.3) and based on nondecreasing and nonincreasing of the function $f(x, y)=\frac{p+x+l y}{x+q y}$.

Let $\left\{y_{n}\right\}_{n=-k}^{\infty}$ be a solution of Eq.( 3.3). Then observe that the following identities are true:

$$
\begin{gather*}
y_{n+1}-1=(q-l)\left[\frac{\frac{p}{q-l}-y_{n-k}}{y_{n}+q y_{n-k}}\right]  \tag{3.17}\\
y_{n+1}-\left(\frac{p+l}{q}\right)=\frac{\left(1-\frac{p+l}{q}\right) y_{n}+p\left(1-y_{n-k}\right)}{y_{n}+q y_{n-k}} \tag{3.18}
\end{gather*}
$$

We will analyze the solution $\left\{y_{n}\right\}_{n=-k}^{\infty}$ under three assumptions:

First, will analyze the solution $\left\{y_{n}\right\}_{n=-k}^{\infty}$ under assumption that

$$
\begin{equation*}
p+l>q \text { and } q>l \tag{3.19}
\end{equation*}
$$

So we have the following consequence which can be resulted directly by using
Eq. ( 3.17) and Eq.(3.18)
Lemma 3.1. Assume that (3.19) holds and let $\left\{y_{n}\right\}_{n=-k}^{\infty}$ be a solution of Eq.(3.3). Then the following statements are true:

1. If for some $N \geq 0, y_{N-k}<\frac{p+l}{q}$. Then $y_{N+1}>1$.
2. If for some $N \geq 0, y_{N-k}=\frac{p+l}{q}$. Then $y_{N+1}=1$.
3. If for some $N \geq 0, y_{N-k}>\frac{p+l}{q}$. Then $y_{N+1}<1$.
4. If for some $N \geq 0, y_{N-k} \geq 1$. Then $y_{N+1} \leq \frac{p+l}{q}$.
5. If for some $N \geq 0, y_{N-k} \leq 1$. Then $y_{N+1} \geq 1$.
6. If for some $N \geq 0,1 \leq y_{N-k} \leq \frac{p+l}{q}$, then $1 \leq y_{N+1} \leq \frac{p+l}{q}$.
7. If for some $N \geq 0,1 \leq y_{N-k}, \cdots, y_{N-1}, y_{N} \leq \frac{p+l}{q}$, then $y_{n} \in\left[1, \frac{p+l}{q}\right]$ for $n \geq N$ that is $\left[1, \frac{p+l}{q}\right]$ is an invariant interval for Eq.(3.3).
8. $1<\bar{y}<\frac{p+l}{q}$.

Indeed: when $p+l>q$
$p+l>q$
$p l+l^{2}>q l$
$p q+p l+l^{2}>l q+p q$
$p q>p q+l q-l p-l^{2}$
$p q>(p+l)(q-l)$
then we have :

$$
\begin{equation*}
\frac{p}{q-l}>\frac{p+l}{q} \tag{3.20}
\end{equation*}
$$

The next result which is a consequence of Theorem 2.9 express that when (3.19) holds, every nontrivial and oscillatory solution of Eq.( 3.3) which lies in the interval $\left[1, \frac{p+l}{q}\right]$ oscillates about equilibrium point $\bar{y}$ with semicycle of length at most $k$.

Theorem 3.5. Assume Eq.( 3.19) holds. Then every nontrivial and oscillatory solution of Eq.(3.3) which lies in the interval $\left[1, \frac{p+l}{q}\right]$ oscillates about $\bar{y}$ with semicycles of length at most $k$.

Second, we will discuss the analysis of semicycles of solution $\left\{y_{n}\right\}_{n=-k}^{\infty}$ under assumption that

$$
\begin{equation*}
p+l<q, q>l \tag{3.21}
\end{equation*}
$$

The following result is a direct consequence of Eq.(3.17) and Eq.(3.18).

Lemma 3.2. Assume 3.21 holds and let $\left\{y_{n}\right\}_{n=-k}^{\infty}$ be a solution of Eq.( 3.3). Then the following statements are true:

1. If for some $N \geq 0, y_{N-k}>\frac{p+l}{q}$ then $y_{N+1}<1$
2. If for some $N \geq 0, y_{N-k}=\frac{p+l}{q}$ then $y_{N+1}=1$
3. If for some $N \geq 0, y_{N-k} \leq 1$ then $y_{N+1}>\frac{p+l}{q}$
4. If for some $N \geq 0, y_{N-k} \leq \frac{p+l}{q}$ then $y_{N+1}>\frac{p+l}{q}$
5. If for some $N \geq 0, \frac{p+l}{q} \leq y_{N-k} \leq 1$ then $\frac{p+l}{q} \leq y_{N+1} \leq 1$
6. If for some $N \geq 0, \frac{p+l}{q} \leq y_{N-k}, \cdots, y_{N-1}, y_{N} \leq 1$, then $y_{n} \in\left[\frac{p+l}{q}, 1\right]$ for $n \geq N$.
7. $\frac{p+l}{q}<\bar{y}<1$.

Indeed: when $p+l<q$
$p+l<q$
$p l+l^{2}<q l$
$p q+p l+l^{2}<l q+p q$
$p q<p q+l q-l p-l^{2}$
$p q<(p+l)(q-l)$
then we have :

$$
\begin{equation*}
\frac{p}{q-l}<\frac{p+l}{q} \tag{3.22}
\end{equation*}
$$

That is $\left[\frac{p+l}{q}, 1\right]$ is an invariant interval for Eq.( 3.3)
The next theorem, which is the result of Theorem 2.10 states that when (3.21) holds, every nontrivial and oscillatory solution of Eq.(3.3) which lies in the interval $\left[\frac{p+l}{q}, 1\right]$ oscillates about equilibrium point $\bar{y}$ with semicycle at of length at least $k$.
Theorem 3.6. Assume Eq.( 3.21) holds. Then every nontrivial and oscillatory solution of Eq.(3.3) which lies in the interval $\left[\frac{p+l}{q}, 1\right]$, after the first semicycle, oscillates about $\bar{y}$ with semicycles of length at least $k$.

Finally, we will analyze the semicycles of solutions $\left\{y_{n}\right\}_{n=-k}^{\infty}$ under assumption that

$$
\begin{equation*}
p+l=q \tag{3.23}
\end{equation*}
$$

In this case Eq.(3.3) will be

$$
\begin{equation*}
y_{n+1}=\frac{p+y_{n}+l y_{n-k}}{y_{n}+(p+l) y_{n-k}} \tag{3.24}
\end{equation*}
$$

with unique equilibrium point $\bar{y}=1$.

Also equations (3.17) and (3.18) are reduced to

$$
\begin{equation*}
y_{n+1}-1=p \frac{1-y_{n-k}}{y_{n}+(p+l) y_{n-k}} \tag{3.25}
\end{equation*}
$$

Lemma 3.3. Let $\left\{y_{n}\right\}_{n=-k}^{\infty}$ be a solution of Eq.( 3.24). Then the following statements are true:

1. If for some $N \geq 0, y_{N-k}<1$, then $y_{N+1}>1$
2. If for some $N \geq 0, y_{N-k}=1$, then $y_{N+1}=1$
3. If for some $N \geq 0, y_{N-k}>1$, then $y_{N+1}<1$

Th next result is direct consequence of lemma 3.3.
Corollary 3.1. Let $\left\{y_{n}\right\}_{n=-k}^{\infty}$ be a solution of Eq. (3.24). Then $\left\{y_{n}\right\}$ oscillates about the equilibrium $\bar{y}=1$.

### 3.8 Global Stability Analysis

Our aim in this section is to establish Global stability of equilibrium point $\bar{y}$ of Eq.( 3.3).

Theorem 3.7. Assume that $q=l$. The equilibrium point $\bar{y}$ of Eq.(3.3) is globally asymptotically stable.

Proof. We proved that equilibrium point $\bar{x}$ is asymptotically stable in Sec. 3.4. Now since $f(u, v)$ is non increasing in both arguments, and solution of the system:

$$
\begin{aligned}
m & =\frac{p+M+q M}{M+q M} \\
M & =\frac{p+m+q m}{m+q m}
\end{aligned}
$$

implies $m=M$. Then by Theorem 2.5, every solution of Eq.(3.3) converges to $\bar{y}$.
Theorem 3.8. Assume that 3.19 holds. Then the equilibrium point $\bar{y}$ of Eq.( 3.3) is globally asymptotically stable in the interval $\left[1, \frac{p+l}{q}\right]$.

Proof. It is enough to show that $\bar{y}$ is global attractor. The condition (3.20) guarantees that $f(u, v)$ is decreasing in both arguments, and solution of the system:

$$
\begin{aligned}
m & =\frac{p+M+l M}{M+q M} \\
M & =\frac{p+m+l m}{m+q m}
\end{aligned}
$$

implies $m=M$. Then by Theorem 2.5, every solution of Eq.(3.3) converges to $\bar{y}$
Theorem 3.9. Assume that 3.21 holds. Then the equilibrium point $\bar{y}$ of Eq.( 3.3) is globally asymptotically stable in the interval $\left[\frac{p+l}{q}, 1\right]$.

Proof. Again it is enough to show that $\bar{y}$ is global attractor. The condition (3.22) guarantees that $f(u, v)$ is increasing in $u$ and decreasing in $v$, and solution of the system:

$$
\begin{aligned}
m & =\frac{p+m+l M}{m+q M} \\
M & =\frac{p+M+l m}{M+q m}
\end{aligned}
$$

implies $m=M$. Then by theorem 2.6, every solution of Eq.( 3.3) converges to $\bar{y}$

## Chapter 4

## The Special Cases $\alpha \beta \gamma B C=0$

In this chapter we examine the character of solution of Eq.(3.1) where one or more of the parameters in Eq.(3.1) are zero. There are many such equations arises by considering one or more parameters are zero.

Observe that some of these equations are meaningless like the case when the parameters in the denominator are zero, and some of them are quite interesting and have been studied by many authors.

### 4.1 One parameter $=0$

In this section we examine the character of solution of Eq. (3.1) where one parameter in Eq.(3.1) equal zero. There are five such equations, namely:

$$
\begin{align*}
& x_{n+1}=\frac{\beta x_{n}+\gamma x_{n-k}}{B x_{n}+C x_{n-k}}, n=0,1,2 \ldots  \tag{4.1}\\
& x_{n+1}=\frac{\alpha+\gamma x_{n-k}}{B x_{n}+C x_{n-k}}, n=0,1,2 \ldots  \tag{4.2}\\
& x_{n+1}=\frac{\alpha+\beta x_{n}}{B x_{n}+C x_{n-k}}, n=0,1,2 \ldots \tag{4.3}
\end{align*}
$$

$$
\begin{align*}
& x_{n+1}=\frac{\alpha+\beta x_{n}+\gamma x_{n-k}}{C x_{n-k}}, n=0,1,2 \ldots  \tag{4.4}\\
& x_{n+1}=\frac{\alpha+\beta x_{n}+\gamma x_{n-k}}{B x_{n}}, n=0,1,2 \ldots \tag{4.5}
\end{align*}
$$

where the parameters $\alpha, \beta, \gamma, B, C$ are nonnegative real numbers and the initial conditions $x_{-k}, x_{-k+1}, \cdots, x_{0}$ are arbitrary nonnegative real numbers.

### 4.1.1 Dynamics of $x_{n+1}=\frac{\beta x_{n}+\gamma x_{n-k}}{B x_{n}+C x_{n-k}}$

The Eq.(4.1) was investigated by Sai'da Abu-baha' in [1].
Lemma 4.1. The change variables $x_{n}=\frac{\gamma}{C} y_{n}$ reduces Eq.(4.1) into the difference equation

$$
\begin{equation*}
y_{n+1}=\frac{p y_{n}+y_{n-k}}{q y_{n}+y_{n-k}}, n=0,1,2 \ldots \tag{4.6}
\end{equation*}
$$

where $p=\frac{\beta}{\gamma}$ and $q=\frac{B}{C}$ with $p, q \in(0, \infty)$ and the initial conditions $y_{-k}, \cdots, y_{0}$ are nonnegative real numbers.

Proof. Substitute $x_{n}=\frac{\gamma}{C} y_{n}$ in Eq.(4.1), we get

$$
\frac{\gamma}{C} y_{n+1}=\frac{\beta \frac{\gamma}{C} y_{n}+\gamma \frac{\gamma}{C} y_{n-k}}{B \frac{\gamma}{C} y_{n}+C \frac{\gamma}{C} y_{n-k}}
$$

then

$$
\frac{\gamma}{C} y_{n+1}=\frac{\beta y_{n}+\gamma y_{n-k}}{B y_{n}+C y_{n-k}}
$$

thus

$$
\frac{\gamma}{C} y_{n+1}=\frac{\gamma\left(\frac{\beta}{\gamma} y_{n}+y_{n-k}\right)}{C\left(\frac{\beta}{C} y_{n}+y_{n-k}\right)}
$$

hence

$$
y_{n+1}=\frac{\frac{\beta}{\gamma} y_{n}+y_{n-k}}{\frac{B}{C} y_{n}+y_{n-k}}
$$

set $p=\frac{\beta}{\gamma}$ and $q=\frac{B}{C}$, we get Eq.(4.6)

She has shown the two cases $p>q$ and $p<q$ give rise to different dynamic behaviors. She examine the investigated of the unique positive equilibrium point
$\bar{y}=\frac{p+1}{q+1}$, period two solution, semicycles, invariant intervals, and global stability. The main results were :

1. when $p>q$

- the equilibrium point is locally asymptotically stable.
- there no period two solution.
- the solution oscillates about equilibrium point $\bar{y}$ with semicycle of length $k+1$ or $k+2$ except possibly for the first semicycle which may have length $k$.
- The solution take its values between 1 and $\frac{p}{q}$.
- The equilibrium point is globally asymptotically stable if $p \leq p q+3 q+1$

2. when $p<q$
(a) k is even.

- the equilibrium point is locally asymptotically stable.
- the solution oscillates about equilibrium point $\bar{y}$ with semisycle of length k after the first semicycle or it converges monotonically to the equilibrium point.
- The solution take its values between $\frac{p}{q}$ and 1 .
(b) k is odd.
i. $q>p q+3 p+1$.
- The equilibrium point is unstable.
- There is a period two solution.
- The solution oscillates about equilibrium point $\bar{y}$ with semisycle of length k after the first semicycle or it converges monotonically to the equilibrium point.
- The solution take its values between $\frac{p}{q}$ and 1 .
ii. $q<p q+3 p+1$
- The equilibrium point is locally asymptotically stable.
- The solution oscillates about equilibrium point $\bar{y}$ with semisycle of length k after the first semicycle or it converges monotonically to the equilibrium point.
- The solution take its values between $\frac{p}{q}$ and 1 .
- The equilibrium point is globally asymptotically stable.


### 4.1.2 Dynamics of $x_{n+1}=\frac{\alpha+\gamma x_{n-k}}{B x_{n}+C x_{n-k}}$

Lemma 4.2. The change variables $x_{n}=\frac{\gamma}{C} y_{n}$ reduces Eq.(4.2) into the difference equation

$$
\begin{equation*}
y_{n+1}=\frac{p+y_{n-k}}{q y_{n}+y_{n-k}}, n=0,1,2 \ldots \tag{4.7}
\end{equation*}
$$

where $p=\frac{\alpha C}{\gamma^{2}}$ and $q=\frac{B}{C}$ with $p, q \in(0, \infty)$ and the initial conditions $y_{-k}, \cdots, y_{0}$ are nonnegative real numbers.

Proof. Substitute $x_{n}=\frac{\gamma}{C} y_{n}$ in Eq.(4.2), we get

$$
\frac{\gamma}{C} y_{n+1}=\frac{\alpha+\gamma \frac{\gamma}{C} y_{n-k}}{B \frac{\gamma}{C} y_{n}+C \frac{\gamma}{C} y_{n-k}}
$$

then

$$
\frac{\gamma}{C} y_{n+1}=\frac{\frac{\alpha C}{\gamma}+\gamma y_{n-k}}{B y_{n}+C y_{n-k}}
$$

thus

$$
\frac{\gamma}{C} y_{n+1}=\frac{\gamma\left(\frac{\alpha C}{\gamma^{2}}+y_{n-k}\right)}{C\left(\frac{B}{C} y_{n}+y_{n-k}\right)}
$$

hence

$$
y_{n+1}=\frac{\frac{\alpha C}{\gamma^{2}}+y_{n-k}}{\frac{B}{C} y_{n}+y_{n-k}}
$$

Set $p=\frac{\alpha C}{\gamma^{2}}$ and $q=\frac{B}{C}$, we get Eq.(4.7).

The Eq.(4.7) was investigated by Devalut, Kosmala, and Ladas in [5].

### 4.1.3 Dynamics of $x_{n+1}=\frac{\alpha+\beta x_{n}}{B x_{n}+C x_{n-k}}$

Lemma 4.3. The Eq.(4.3) is reduced by the change variables $x_{n}=\frac{\beta}{B} y_{n}$ into the

$$
\begin{equation*}
y_{n+1}=\frac{p+y_{n}}{y_{n}+q y_{n-k}}, n=0,1,2 \ldots \tag{4.8}
\end{equation*}
$$

where $p=\frac{\alpha B}{\beta^{2}}$ and $q=\frac{C}{B}$ with $p, q \in(0, \infty)$ and the initial conditions $y_{-k}, \cdots, y_{0}$ are nonnegative real numbers.
Proof. Substitute $x_{n}=\frac{\beta}{B} y_{n}$ in Eq.(4.3), we get

$$
\frac{\beta}{B} y_{n+1}=\frac{\alpha+\beta \frac{\beta}{B} y_{n}}{B \frac{\beta}{B} y_{n}+C \frac{\beta}{B} y_{n-k}}
$$

then

$$
\frac{\beta}{B} y_{n+1}=\frac{\frac{\alpha B}{\beta}+\beta y_{n}}{B y_{n}+C y_{n-k}}
$$

thus

$$
\frac{\beta}{B} y_{n+1}=\frac{\beta\left(\frac{\alpha B}{\beta^{2}}+y_{n}\right)}{B\left(y_{n}+\frac{C}{B} y_{n-k}\right)}
$$

hence

$$
y_{n+1}=\frac{\frac{\alpha B}{\beta^{2}}+y_{n}}{y_{n}+\frac{C}{B} y_{n-k}}
$$

Set $p=\frac{\alpha B}{\beta^{2}}$ and $q=\frac{C}{B}$, we get Eq.(4.8).

The Eq.(4.8) was investigated in [3]. They have concentrated on invariant intervals, the character of semicycles, the global stability, and the boundedness.

### 4.1.4 Dynamics of $x_{n+1}=\frac{\alpha+\beta x_{n}+\gamma x_{n-k}}{C x_{n-k}}$

Lemma 4.4. The Eq.(4.4) is reduced by change of variable $x_{n}=\frac{\gamma}{C} y_{n}+\frac{\gamma}{C}$, into the difference equation

$$
\begin{equation*}
y_{n+1}=\frac{p+q y_{n}}{1+y_{n-k}}, n=0,1,2 \ldots \tag{4.9}
\end{equation*}
$$

where $p=\frac{\alpha C+\beta \gamma}{\gamma^{2}}$ and $q=\frac{\beta}{\gamma}$ with $p, q \in(0, \infty)$ and the initial conditions $y_{-k}, \cdots, y_{0}$ are nonnegative real numbers.

Proof. Substitute $x_{n}=\frac{\gamma}{C} y_{n}+\frac{\gamma}{C}$ in Eq.(4.4). We get

$$
\frac{\gamma}{C} y_{n+1}+\frac{\gamma}{C}=\frac{\alpha+\beta\left(\frac{\gamma}{C} y_{n}+\frac{\gamma}{C}\right)+\gamma\left(\frac{\gamma}{C} y_{n-k}+\frac{\gamma}{C}\right)}{C\left(\frac{\gamma}{C} y_{n-k}+\frac{\gamma}{C}\right)}
$$

then

$$
\frac{\gamma}{C} y_{n+1}=\frac{\alpha+\beta\left(\frac{\gamma}{C} y_{n}+\frac{\gamma}{C}\right)+\gamma\left(\frac{\gamma}{C} y_{n-k}+\frac{\gamma}{C}\right)}{C\left(\frac{\gamma}{C} y_{n-k}+\frac{\gamma}{C}\right)}-\frac{\gamma}{C}
$$

eliminate $C$ in the denominators

$$
\gamma y_{n+1}=\frac{\alpha+\beta\left(\frac{\gamma}{C} y_{n}+\frac{\gamma}{C}\right)+\gamma\left(\frac{\gamma}{C} y_{n-k}+\frac{\gamma}{C}\right)}{\frac{\gamma}{C} y_{n-k}+\frac{\gamma}{C}}-\frac{\gamma}{1}
$$

thus

$$
\gamma y_{n+1}=\frac{\alpha+\beta\left(\frac{\gamma}{C} y_{n}+\frac{\gamma}{C}\right)+\gamma\left(\frac{\gamma}{C} y_{n-k}+\frac{\gamma}{C}\right)-\gamma\left(\frac{\gamma}{C} y_{n-k}+\frac{\gamma}{C}\right)}{\frac{\gamma}{C} y_{n-k}+\frac{\gamma}{C}}
$$

then

$$
\gamma y_{n+1}=\frac{\alpha+\beta\left(\frac{\gamma}{C} y_{n}+\frac{\gamma}{C}\right)}{\frac{\gamma}{C} y_{n-k}+\frac{\gamma}{C}}
$$

then

$$
\gamma y_{n+1}=\frac{\frac{\alpha C}{\gamma}+\beta y_{n}+\beta}{y_{n-k}+1}
$$

then

$$
\gamma y_{n+1}=\frac{\gamma\left(\frac{\alpha C}{\gamma^{2}}+\frac{\beta}{\gamma}+\frac{\beta}{\gamma} y_{n}\right)}{1+y_{n-k}}
$$

hence

$$
y_{n+1}=\frac{\frac{\alpha C}{\gamma^{2}}+\frac{\beta}{\gamma}+\frac{\beta}{\gamma} y_{n}}{1+y_{n-k}}
$$

Set $p=\frac{\alpha C}{\gamma^{2}}+\frac{\beta}{\gamma}=\frac{\alpha C+\beta \gamma}{\gamma^{2}}$ and $q=\frac{\beta}{\gamma}$, we get Eq.(4.9).

The Eq.(4.9) was investigated in [4]. The authors studied the global stability, boundedness of positive solutions, and character of semicycles of Eq.(4.9).

### 4.1.5 Dynamics of $x_{n+1}=\frac{\alpha+\beta x_{n}+\gamma x_{n-k}}{B x_{n}}$

Lemma 4.5. The Eq.(4.5) is reduced by change of variable $x_{n}=\frac{\beta}{B} y_{n}+\frac{\beta}{B}$, into

$$
\begin{equation*}
y_{n+1}=\frac{p+q y_{n-k}}{1+y_{n}}, n=0,1,2 \ldots \tag{4.10}
\end{equation*}
$$

where $p=\frac{\alpha B+C \beta}{\beta^{2}}$ and $q=\frac{\gamma}{\beta}$ with $p, q \in(0, \infty)$ and the initial conditions $y_{-k}, \cdots, y_{0}$ are nonnegative real numbers.

Proof. The proof of this case is similar to Eq.(4.4) and can be omitted.

The Eq.(4.10) was investigated in [10] by Mahdi Dehghan, M. Jaberi Douraki, and M. Razzaghi.

### 4.2 Two parameters are zero

In this section we examine the character of solution of Eq. (3.1) where two parameters in Eq.(3.1) are zero. There are eight such equations, namely:

$$
\begin{equation*}
x_{n+1}=\frac{\gamma x_{n-k}}{B x_{n}+C x_{n-k}}, n=0,1,2 \ldots \tag{4.11}
\end{equation*}
$$

$$
\begin{gather*}
x_{n+1}=\frac{\beta x_{n}}{B x_{n}+C x_{n-k}}, n=0,1,2 \ldots  \tag{4.12}\\
x_{n+1}=\frac{\beta x_{n}+\gamma x_{n-k}}{C x_{n-k}}, n=0,1,2 \ldots  \tag{4.13}\\
x_{n+1}=\frac{\beta x_{n}+\gamma x_{n-k}}{B x_{n}}, n=0,1,2 \ldots  \tag{4.14}\\
x_{n+1}=\frac{\alpha}{B x_{n}+C x_{n-k}}, n=0,1,2 \ldots  \tag{4.15}\\
x_{n+1}=\frac{\alpha+\gamma x_{n-k}}{B x_{n}}, n=0,1,2 \ldots  \tag{4.16}\\
x_{n+1}=\frac{\alpha+\beta x_{n}}{C x_{n-k}}, n=0,1,2 \ldots  \tag{4.17}\\
x_{n+1}=\frac{\alpha+\beta x_{n}}{B x_{n}}, n=0,1,2 \ldots \tag{4.18}
\end{gather*}
$$

where the parameters $\alpha, \beta, \gamma, B, C$ are nonnegative real numbers and the initial conditions $x_{-k}, x_{-k+1}, \cdots, x_{0}$ are arbitrary nonnegative real numbers.

### 4.2.1 $\quad$ Dynamics of $x_{n+1}=\frac{\gamma x_{n-k}}{B x_{n}+C x_{n-k}}$

Lemma 4.6. The Eq.(4.11) is reduced by the change of variables $x_{n}=\frac{\gamma}{B y_{n}}$ into

$$
\begin{equation*}
y_{n+1}=P+\frac{y_{n-k}}{y_{n}}, n=0,1,2, \ldots \tag{4.19}
\end{equation*}
$$

where $P=\frac{C}{B} \in(0, \infty)$ and the initial conditions $y_{-k}, \cdots, y_{0}$ are nonnegative real numbers.

Proof. Substitute $x_{n}=\frac{\gamma}{B y_{n}}$ in Eq.(4.11). We get

$$
\frac{\gamma}{B y_{n+1}}=\frac{\frac{\gamma^{2}}{B y_{n-k}}}{\frac{\gamma}{y_{n}}+\frac{C \gamma}{B y_{n-k}}}
$$

cancel $\frac{\gamma}{B}$ from two sides, we get

$$
\frac{1}{y_{n+1}}=\frac{\frac{\gamma}{y_{n-k}}}{\frac{\gamma}{y_{n}}+\frac{C \gamma}{B y_{n-k}}}
$$

then

$$
\frac{1}{y_{n+1}}=\frac{\frac{1}{y_{n-k}}}{\frac{1}{y_{n}}+\frac{C}{B y_{n-k}}}
$$

by taking reciprocal of both sides

$$
y_{n+1}=\frac{\frac{1}{y_{n}}+\frac{C}{B y_{n-k}}}{\frac{1}{y_{n-k}}}
$$

hence

$$
y_{n+1}=\frac{y_{n-k}}{y_{n}}+\frac{C}{B}
$$

set $P=\frac{C}{B}$, we get Eq.(4.19). The proof is complete

The Eq.(4.19) was investigated in [13] and [9]. But in [9], the investigation is restricted for $P \in[1, \infty)$.

### 4.2.2 Dynamics of $x_{n+1}=\frac{\beta x_{n}}{B x_{n}+C x_{n-k}}$

Lemma 4.7. The change of variables $x_{n}=\frac{\beta}{C y_{n}}$ reduces Eq.(4.12) into the difference equation

$$
\begin{equation*}
y_{n+1}=P+\frac{y_{n}}{y_{n-k}}, n=0,1,2, \ldots \tag{4.20}
\end{equation*}
$$

where $P=\frac{B}{C}$ and the initial conditions $y_{-k}, \cdots, y_{0}$ are nonnegative real numbers.
Proof. Substitute $x_{n}=\frac{\beta}{C y_{n}}$ in Eq.(4.12). We get

$$
\frac{\beta}{C y_{n+1}}=\frac{\frac{\beta^{2}}{C y_{n}}}{\frac{B \beta}{C y_{n}}+\frac{\beta}{y_{n-k}}}
$$

cancel $\frac{\beta}{C}$ from both sides, we get

$$
\frac{1}{y_{n+1}}=\frac{\frac{\beta}{y_{n}}}{\frac{B \beta}{C y_{n}}+\frac{\beta}{y_{n-k}}}
$$

then by canceling $\beta$, we get

$$
\frac{1}{y_{n+1}}=\frac{\frac{1}{y_{n}}}{\frac{B}{C y_{n}}+\frac{1}{y_{n-k}}}
$$

by taking reciprocal of both sides

$$
y_{n+1}=\frac{\frac{B}{C y_{n}}+\frac{1}{y_{n-k}}}{\frac{1}{y_{n}}}
$$

thus

$$
y_{n+1}=\frac{B}{C}+\frac{y_{n}}{y_{n-k}}
$$

set $P=\frac{B}{C}$, we get the Eq.(4.20). This completes the proof.

The Eq.(4.20) was studied in [2].

### 4.2.3 Dynamics of $x_{n+1}=\frac{\beta x_{n}+\gamma x_{n-k}}{C x_{n-k}}$

Lemma 4.8. The change of variables $x_{n}=\frac{\beta}{C} y_{n}$ reduces The Eq.(4.13) into the difference equation

$$
\begin{equation*}
y_{n+1}=P+\frac{y_{n}}{y_{n-k}}, n=0,1,2, \ldots \tag{4.21}
\end{equation*}
$$

where $P=\frac{\gamma}{\beta}$ and the initial conditions $y_{-k}, \cdots, y_{0}$ are nonnegative real numbers.
Proof. Substitute $x_{n}=\frac{\beta}{C}$ in Eq.(4.13). We get

$$
\frac{\beta}{C} y_{n+1}=\frac{\frac{\beta}{C} \beta y_{n}+\frac{\beta}{C} \gamma y_{n-k}}{\frac{\beta}{C} C y_{n-k}}
$$

hence

$$
\frac{\beta}{C} y_{n+1}=\frac{\beta y_{n}+\gamma y_{n-k}}{C y_{n-k}}
$$

then

$$
\frac{\beta}{C} y_{n+1}=\frac{\beta\left[y_{n}+\frac{\gamma}{\beta} y_{n-k}\right]}{C y_{n-k}}
$$

by eliminating $\frac{\beta}{C}$ from both side, we get

$$
y_{n+1}=\frac{y_{n}+\frac{\gamma}{\beta} y_{n-k}}{y_{n-k}}
$$

hence

$$
y_{n+1}=\frac{y_{n}}{y_{n-k}}+\frac{\gamma}{\beta}
$$

set $p=\frac{\gamma}{\beta}$, we get Eq.(4.21). The proof is complete.

The Eq.(4.20) was studied in [2].

### 4.2.4 Dynamics of $x_{n+1}=\frac{\beta x_{n}+\gamma x_{n-k}}{B x_{n}}$

Lemma 4.9. The change of variables $x_{n}=\frac{\gamma}{B} y_{n}$ reduces Eq.(4.14) to the difference equation

$$
\begin{equation*}
y_{n+1}=P+\frac{y_{n-k}}{y_{n}}, n=0,1,2, \ldots \tag{4.22}
\end{equation*}
$$

where $P=\frac{\beta}{\gamma} \in(0, \infty)$ and the initial conditions $y_{-k}, \cdots, y_{0}$ are nonnegative real numbers.

Proof. Substitute $x_{n}=\frac{\gamma}{B}$ in Eq.(4.14). We get

$$
\frac{\gamma}{B} y_{n+1}=\frac{\frac{\gamma}{B} \beta y_{n}+\frac{\gamma}{B} \gamma y_{n-k}}{\frac{\gamma}{B} B y_{n}}
$$

hence

$$
\frac{\gamma}{B} y_{n+1}=\frac{\beta y_{n}+\gamma y_{n-k}}{B y_{n}}
$$

then

$$
\frac{\gamma}{B} y_{n+1}=\frac{\gamma\left[\frac{\beta}{\gamma} y_{n}+y_{n-k}\right]}{B y_{n}}
$$

by eliminating $\frac{\gamma}{B}$ from both side, we get

$$
y_{n+1}=\frac{\frac{\beta}{\gamma} y_{n}+y_{n-k}}{y_{n}}
$$

hence

$$
y_{n+1}=\frac{\beta}{\gamma}+\frac{y_{n-k}}{y_{n}}
$$

set $P=\frac{\beta}{\gamma}$, we get Eq.(4.22). The proof is complete.
The Eq.(4.22) was investigated in [13] and [9]. But in [9], the investigation is restricted for $P \in[1, \infty)$.

### 4.2.5 Dynamics of $x_{n+1}=\frac{\alpha}{B x_{n}+C x_{n-k}}$

The Eq.(4.15) is reduced by change of variables $x_{n}=\frac{\sqrt{\alpha}}{y_{n}}$ into

$$
\begin{equation*}
y_{n+1}=\frac{B}{y_{n}}+\frac{C}{y_{n-k}}, n=0,1,2, \ldots \tag{4.23}
\end{equation*}
$$

where the initial conditions $y_{-k}, \cdots, y_{0}$ are arbitrary nonnegative real numbers.
The only positive equilibrium point is $\bar{y}=\sqrt{B+C}$. when $k=1$, the Eq.(4.23) was investigated in [11]. It was shown that every solution is bounded and persists, it also shown that the equilibrium point $\bar{y}=\sqrt{B+C}$ is globally asymptotically stable. In this monograph, we investigate the difference equation (4.23) when $k \in\{2,3, \ldots\}$.

Theorem 4.1. Every solution of Eq.(4.23) is bounded and persists.
Proof. Let the contrary. i.e. there exists a solution $\left\{y_{n}\right\}_{n=-k}^{\infty}$ which is neither bounded from above nor from below. That is

$$
\lim _{n \rightarrow \infty} \sup y_{n}=\infty \text { and } \lim _{n \rightarrow \infty} \inf y_{n}=0
$$

Then clearly, we can find indices i and j with

$$
1 \leq i<j
$$

such that

$$
y_{i}>y_{n}>y_{j} \text { for all } n \in\{-k, \ldots, j-1\}
$$

Hence

$$
y_{j}=\frac{B}{y_{j-1}}+\frac{C}{y_{j-k-1}}>\frac{B+C}{y_{i}}
$$

and

$$
y_{i}=\frac{B}{y_{i-1}}+\frac{C}{y_{i-k-1}} \leq \frac{B+C}{y_{j}}
$$

that is

$$
B+C<y_{i} y_{j}<B+C
$$

which is impossible.

To investigate the stability of Eq.(4.23), let $f(x, y)=\frac{B}{x}+\frac{C}{y}$.

Theorem 4.2. The equilibrium point $\bar{y}=\sqrt{B+C}$ is unstable when $k$ is even.

Proof. The linearized equation of Eq.(4.23) about the equilibrium point $\bar{y}=\sqrt{B+C}$ is

$$
z_{n+1}=-\frac{B}{B+C} z_{n}-\frac{C}{B+C} z_{n-k}, n=0,1,2, \ldots
$$

and its characteristic equation is

$$
\lambda^{k+1}+\frac{B}{B+C} \lambda^{k}+\frac{C}{B+C}=0
$$

Then the proof follows immediately from theorem 2.3 .

Before we examine the existence of two cycles of eq.(4.23), it is worthwhile to mention that when $C=1$ and $k=2$, it was shown by R.Devault and G. Ladas and S.W. Schultz that every positive solution of the difference equation $y_{n+1}=\frac{B}{y_{n}}+\frac{1}{y_{n-2}}$ converges to a period two solution.

Theorem 4.3. Let $\left\{y_{n}\right\}_{n=-k}^{\infty}$ be a non-negative solution of Eq.(4.23). Then the following statements are true:

- If $k$ is odd, then Eq.(4.23) does not have prime period-two solutions.
- If $k$ is even, then Eq.(4.23) has prime period-two solution

$$
\ldots, \phi, \psi, \phi, \psi, \ldots
$$

And the values $\phi$ and $\psi$ of all prime period-two solutions are given by:

$$
\{\phi, \psi \in(0, \infty): \phi \psi=B+C\}
$$

Proof. Let

$$
\ldots, \Phi, \Psi, \Phi, \Psi, \ldots
$$

be a period two solution of the Eq.(4.23) where $\psi$ and $\phi$ are positive and distinctive, then

- If k is odd, then we have

$$
\begin{align*}
& \Phi=\frac{B}{\Psi}+\frac{C}{\Phi}  \tag{4.24}\\
& \psi=\frac{B}{\Phi}+\frac{C}{\Psi} \tag{4.25}
\end{align*}
$$

from Eq.(4.24), we get

$$
\begin{equation*}
\Phi=\frac{B \phi+C \psi}{\phi \psi} \tag{4.26}
\end{equation*}
$$

from Eq.(4.25), we get

$$
\begin{equation*}
\psi=\frac{B \psi+C \phi}{\phi \psi} \tag{4.27}
\end{equation*}
$$

from Eq.(4.26) and Eq.(4.27), we get

$$
\begin{equation*}
\frac{B \phi+C \psi}{\phi}=\frac{B \psi+C \phi}{\psi} \tag{4.28}
\end{equation*}
$$

then

$$
\begin{equation*}
B \phi \psi+C \psi^{2}=B \psi \phi+C \phi^{2} \tag{4.29}
\end{equation*}
$$

hence

$$
C \psi^{2}=B \phi^{2}
$$

thus

$$
\phi=\psi
$$

which is contradiction.

- If k is even, then

$$
\Phi=\frac{B}{\Psi}+\frac{C}{\Psi} \text { and } \psi=\frac{B}{\Phi}+\frac{C}{\Phi}
$$

which implies that

$$
\phi \psi=B+C
$$

and the period two solution must be of the form

$$
\ldots, \phi, \frac{B+C}{\phi}, \phi, \frac{B+C}{\phi}, \ldots
$$

which is completes the proof.

Theorem 4.4. Let $\left\{y_{n}\right\}_{n=-k}^{\infty}$ be a solution of Eq.(4.23). Then the following statements are true:

1. Suppose $B+C>1$ and assume that for some $N \geq 0, y_{N-k}, \ldots, y_{N-1}, y_{N} \in$ $[1, B+C]$. Then $y_{n} \in[1, B+C]$ for all $n>N$.
2. Suppose $B+C<1$ and assume that for some $N \geq 0, y_{N-k}, \ldots, y_{N-1}, y_{N} \in$ $[B+C, 1]$. Then $y_{n} \in[B+C, 1]$ for all $n>N$.
3. Suppose $B>C$ and assume that for some $N \geq 0, y_{N-k}, \ldots, y_{N-1}, y_{N} \in$ $\left[C, \frac{B}{C}+1\right]$. Then $y_{n} \in\left[C, \frac{B}{C}+1\right]$ for all $n>N$.
4. Suppose $B<C$ and assume that for some $N \geq 0, y_{N-k}, \ldots, y_{N-1}, y_{N} \in$ $\left[B, \frac{C}{B}+1\right]$. Then $y_{n} \in\left[B, \frac{C}{B}+1\right]$ for all $n>N$.

Proof. The proof of this theorem is based on monotonic character.

1. Assume that for some $N>0, y_{N-k}, \ldots, y_{N-1}, y_{N} \in[1, B+C]$. Then

$$
y_{N+1}=\frac{B}{y_{N}}+\frac{C}{y_{N-k}} \leq B+C
$$

and

$$
y_{N+1}=\frac{B}{y_{N}}+\frac{C}{y_{N-k}} \geq \frac{B}{B+C}+\frac{C}{B+C}=1
$$

2. Assume that for some $N>0, y_{N-k}, \ldots, y_{N-1}, y_{N} \in[B+C, 1]$. Then

$$
y_{N+1}=\frac{B}{y_{N}}+\frac{C}{y_{N-k}} \leq \frac{B}{B+C}+\frac{C}{B+C}=1
$$

and

$$
y_{N+1}=\frac{B}{y_{N}}+\frac{C}{y_{N-k}} \geq B+C
$$

3. Assume that for some $N>0, y_{N-k}, \ldots, y_{N-1}, y_{N} \in\left[C, \frac{B}{C}+1\right]$. Then

$$
y_{N+1}=\frac{B}{y_{N}}+\frac{C}{y_{N-k}} \leq \frac{B}{C}+\frac{C}{C}=\frac{B}{C}+1
$$

and

$$
y_{N+1}=\frac{B}{y_{N}}+\frac{C}{y_{N-k}} \geq \frac{B}{\frac{B+C}{C}}+\frac{C}{\frac{B+C}{C}}=C
$$

4. Assume that for some $N \geq 0, y_{N-k}, \ldots, y_{N-1}, y_{N} \in\left[B, \frac{C}{B}+1\right]$. Then

$$
y_{N+1}=\frac{B}{y_{N}}+\frac{C}{y_{N-k}} \leq \frac{B}{B}+\frac{C}{B}=\frac{C}{B}+1
$$

and

$$
y_{N+1}=\frac{B}{y_{N}}+\frac{C}{y_{N-k}} \geq \frac{B}{\frac{B+C}{B}}+\frac{C}{\frac{B+C}{B}}=B
$$

The proof is complete.
Theorem 4.5. Let $k$ is odd, Then $\bar{y}=\sqrt{B+C}$ is globally attractor equilibrium point of Eq.(4.23).

Proof. For $u, v \in(0, \infty)$, set

$$
f(u, v)=\frac{B}{u}+\frac{C}{v}
$$

Then $f:(0, \infty) \times(0, \infty) \rightarrow(0, \infty)$ is continuous function and is nonincreasing in both its argument. Let $(m, M) \in(0, \infty)$ is a solution of the system

$$
m=f(M, M) \text { and } M=f(m, m)
$$

then $m=M$ when $\mathbf{k}$ is odd. By using Theorem $2.5, \bar{y}=\sqrt{B+C}$ is globally asymptotically stable equilibrium point of Eq.(4.23). This completes the proof.

Finally, we introduce the analysis od semicycles of Eq.(4.23) in the following theorem.

Theorem 4.6. Every oscillatory solution of Eq.(4.23) has semicycle of length at most $k$

Proof. The proof follows from theorem 2.9 by observing that the function $f(u, v)=$ $\frac{B}{u}+\frac{C}{v}$ is decreasing in both its arguments. The proof is complete.

### 4.2.6 Dynamics of $x_{n+1}=\frac{\alpha+\gamma x_{n-k}}{B x_{n}}$

When $\beta=C=0$ we get the Eq.(4.16)
Lemma 4.10. The change of variables $x_{n}=\frac{\gamma}{B} y_{n}$ reduces Eq.(4.16) into the difference equation

$$
\begin{equation*}
y_{n+1}=\frac{P+y_{n-k}}{y_{n}} \tag{4.30}
\end{equation*}
$$

where $P=\frac{\alpha B}{\gamma^{2}} \in(0, \infty)$ and the initial conditions $y_{-k}, \cdots, y_{0}$ are arbitrary nonnegative real numbers.

Proof. Substitute $x_{n}=\frac{\gamma}{B} y_{n}$ in Eq.(4.16) to get

$$
\frac{\gamma}{B} y_{n+1}=\frac{\alpha+\gamma \frac{\gamma}{B} y_{n-k}}{B \frac{\gamma}{B} y_{n}}
$$

then

$$
\frac{\gamma}{B} y_{n+1}=\frac{\frac{\alpha B}{\gamma}+\gamma y_{n-k}}{B y_{n}}
$$

thus

$$
\frac{\gamma}{B} y_{n+1}=\frac{\gamma\left[\frac{\alpha B}{\gamma^{2}}+y_{n-k}\right]}{B y_{n}}
$$

by canceling $\frac{\gamma}{B}$ from both sides, we get

$$
y_{n+1}=\frac{\frac{\alpha B}{\gamma^{2}}+y_{n-k}}{y_{n}}
$$

set $P=\frac{\alpha B}{\gamma^{2}}$, we get Eq.(4.30)

The only positive equilibrium point is $\bar{y}=\frac{1+\sqrt{1+4 p}}{2}$. The linearized equation about equilibrium point $\bar{y}$ is

$$
z_{n+1}+z_{n}-\frac{2}{1+\sqrt{1+4 p}} z_{n-k}=0
$$

and its characteristic equation is:

$$
\lambda^{k+1}+\lambda^{k}-\frac{2}{1+\sqrt{1+4 p}} .
$$

Theorem 4.7. The equilibrium point $\bar{y}=\frac{1+\sqrt{1+4 p}}{2}$ is unstable.

The proof follow immediately by Theorem 2.3.
Theorem 4.8. The Eq.(4.30) has no positive prime period two solution.
Proof. Let there exist a solution of prime period two

$$
\ldots, \phi, \psi, \phi, \psi, \ldots
$$

where $\phi$ and $\psi$ are positive and distinct.

- If k is odd. Then we have

$$
\begin{equation*}
\phi=\frac{p+\phi}{\psi} \tag{4.31}
\end{equation*}
$$

and

$$
\begin{equation*}
\psi=\frac{p+\psi}{\phi} \tag{4.32}
\end{equation*}
$$

from Eq.(4.31), we get

$$
\begin{equation*}
\phi \psi=p+\phi \tag{4.33}
\end{equation*}
$$

and from Eq.(4.32), we get

$$
\begin{equation*}
\phi \psi=p+\psi \tag{4.34}
\end{equation*}
$$

from Eq.(4.33)and Eq.(4.34), we get

$$
p+\phi=p+\psi
$$

Hence

$$
\psi=\phi
$$

which is a contradiction.

- If k is even. Then we have

$$
\begin{equation*}
\phi=\frac{p+\psi}{\psi} \tag{4.35}
\end{equation*}
$$

and

$$
\begin{equation*}
\psi=\frac{p+\phi}{\phi} \tag{4.36}
\end{equation*}
$$

from Eq.(4.35), we get

$$
\begin{equation*}
\phi \psi=p+\psi \tag{4.37}
\end{equation*}
$$

and from Eq.(4.36), we get

$$
\begin{equation*}
\phi \psi=p+\phi \tag{4.38}
\end{equation*}
$$

from Eq.(4.46)and Eq.(4.47), we get

$$
p+\phi=p+\psi
$$

Hence

$$
\psi=\phi
$$

which is a contradiction.
This completes the proof.
Theorem 4.9. The equilibrium point $\bar{y}=\frac{1+\sqrt{1+4 p}}{2}$ of Eq.(4.30) is global attractor.

Proof. For $u, v \in(0, \infty)$, set $f(u, v)=\frac{p+v}{u}$. Then $f:(0, \infty) \times(0, \infty) \rightarrow(0, \infty)$ is continuous function and is nonincreasing in $u$ and nondecreasing in $v$. Let $(m, M) \in$ $(0, \infty)$ is a solution of the system

$$
m=f(M, m) \text { and } M=f(m, M)
$$

Then

$$
p+m=p+M
$$

Hence

$$
m=M
$$

Then by using theorem 2.7, $\bar{y}=\frac{1+\sqrt{1+4 p}}{2}$ is a global attractor equilibrium point of Eq.(4.30). This completes the proof.

Theorem 4.10. Every oscillatory solution of Eq.(4.30) has semisycle of length $k$.
Proof. The proof follows immediately from theorem 2.8 by observing that the function $f(x, y)=\frac{p+y}{x}$ is decreasing in $x$ and increasing in $y$. The proof is complete.

### 4.2.7 Dynamics of $x_{n+1}=\frac{\alpha+\beta x_{n}}{C x_{n-k}}$

Lemma 4.11. The change of variables $x_{n}=\frac{\beta}{C} y_{n}$ reduces the Eq.(4.17) to the difference equation

$$
\begin{equation*}
y_{n+1}=\frac{P+y_{n}}{y_{n-k}} \tag{4.39}
\end{equation*}
$$

where $P=\frac{\alpha C}{\beta^{2}} \in(0, \infty)$ and the initial conditions $y_{-k}, \cdots, y_{0}$ are arbitrary nonnegative real numbers.

Proof. Substitute $x_{n}=\frac{\beta}{C} y_{n}$ in Eq.(4.17) to get

$$
\frac{\beta}{C} y_{n+1}=\frac{\alpha+\beta \frac{\beta}{C} y_{n}}{C \frac{\beta}{C} y_{n-k}}
$$

then

$$
\frac{\beta}{C} y_{n+1}=\frac{\frac{\alpha C}{\beta}+\beta y_{n}}{C y_{n-k}}
$$

thus

$$
\frac{\beta}{C} y_{n+1}=\frac{\beta\left[\frac{\alpha C}{\beta^{2}}+y_{n}\right]}{C y_{n-k}}
$$

by canceling $\frac{\beta}{C}$ from both sides, we get

$$
y_{n+1}=\frac{\frac{\alpha C}{\beta^{2}}+y_{n}}{y_{n-k}}
$$

set $P=\frac{\alpha C}{\beta^{2}}$, we get Eq.(4.39)

The only positive equilibrium point is $\bar{y}=\frac{1+\sqrt{1+4 p}}{2}$. The linearized equation about equilibrium point $\bar{y}$ is

$$
z_{n+1}-\frac{2}{1+\sqrt{1+4 p}} z_{n}+z_{n-k}=0
$$

and its characteristic equation is:

$$
\lambda^{k+1}-\frac{2}{1+\sqrt{1+4 p}} \lambda^{k}+1=0
$$

Remark 4.1. For $k=1$, the Eq.(4.17) is well known in literature of Rational Difference Equations as lyness' Equation [11]. For this equation it is known that every solution is bounded and persists and no nontrivial solution had a limit.

Theorem 4.11. The equilibrium point $\bar{y}=\frac{1+\sqrt{1+4 p}}{2}$ is unstable.

The proof follow immediately by Theorem 2.3.
Theorem 4.12. The Eq.(4.39) has no positive prime period two solution.
Proof. Let there exist a solution of prime period two

$$
\ldots, \phi, \psi, \phi, \psi, \ldots
$$

where $\phi$ and $\psi$ are positive and distinct.

- If k is odd. Then we have

$$
\begin{equation*}
\phi=\frac{p+\psi}{\phi} \tag{4.40}
\end{equation*}
$$

and

$$
\begin{equation*}
\psi=\frac{p+\phi}{\psi} \tag{4.41}
\end{equation*}
$$

from Eq.(4.40), we get

$$
\begin{equation*}
\phi^{2}=p+\psi \tag{4.42}
\end{equation*}
$$

and from Eq.(4.41), we get

$$
\begin{equation*}
\psi^{2}=p+\phi \tag{4.43}
\end{equation*}
$$

from Eq.(4.42) and Eq.(4.43), we get

$$
\phi+\psi=-1
$$

Which is a contradiction.

- If k is even. Then we have

$$
\begin{equation*}
\phi=\frac{p+\psi}{\psi} \tag{4.44}
\end{equation*}
$$

and

$$
\begin{equation*}
\psi=\frac{p+\phi}{\phi} \tag{4.45}
\end{equation*}
$$

from Eq.(4.44), we get

$$
\begin{equation*}
\phi \psi=p+\psi \tag{4.46}
\end{equation*}
$$

and from Eq.(4.45), we get

$$
\begin{equation*}
\phi \psi=p+\phi \tag{4.47}
\end{equation*}
$$

from Eq.(4.46) and Eq.(4.47), we get

$$
p+\phi=p+\psi
$$

Hence

$$
\psi=\phi
$$

which is a contradiction.
This completes the proof.
Theorem 4.13. The equilibrium point $\bar{y}=\frac{1+\sqrt{1+4 p}}{2}$ of $E q$.(4.39) is global attractor when $k$ is even.

Proof. For $u, v \in(0, \infty)$, set $f(u, v)=\frac{p+u}{v}$. Then $f:(0, \infty) \times(0, \infty) \rightarrow(0, \infty)$ is continuous function and is nondecreasing in $u$ and nonincreasing in $v$. Let $(m, M) \in$ $(0, \infty)$ is a solution of the system

$$
m=f(m, M) \text { and } M=f(M, m)
$$

Then

$$
p+m=p+M
$$

Hence

$$
m=M
$$

Then by using theorem 2.6, $\bar{y}=\frac{1+\sqrt{1+4 p}}{2}$ is a global attractor equilibrium point of Eq.(4.39). This completes the proof.

Theorem 4.14. Every oscillatory solution of Eq.(4.39) has semisycle of length at least $k+1$.

Proof. The proof follows immediately from theorem 2.10 by observing that the function $f(x, y)=\frac{p+x}{y}$ is increasing in $x$ and decreasing in $y$. The proof is complete.

The Eq.(4.18) is a Riccati equation and can be solved explicitly to determine the character of its solution [6]. And the equilibrium point is globally asymptotically stable.

### 4.3 Three parameters are zero

In this section we examine the character of solution of Eq. (3.1) where three parameters in Eq.(3.1) are zero. There are six such equations, namely:

$$
\begin{align*}
& x_{n+1}=\frac{\gamma x_{n-k}}{C x_{n-k}}, n=0,1,2 \ldots  \tag{4.48}\\
& x_{n+1}=\frac{\gamma x_{n-k}}{B x_{n}}, n=0,1,2 \ldots  \tag{4.49}\\
& x_{n+1}=\frac{\beta x_{n}}{C x_{n-k}}, n=0,1,2 \ldots \tag{4.50}
\end{align*}
$$

$$
\begin{equation*}
x_{n+1}=\frac{\beta x_{n}}{B x_{n}}, n=0,1,2 \ldots \tag{4.51}
\end{equation*}
$$

$$
\begin{equation*}
x_{n+1}=\frac{\alpha}{C x_{n-k}}, n=0,1,2 \ldots \tag{4.52}
\end{equation*}
$$

$$
\begin{equation*}
x_{n+1}=\frac{\alpha}{B x_{n}}, n=0,1,2 \ldots \tag{4.53}
\end{equation*}
$$

where the parameters $\alpha, \beta, \gamma, B, C$ are nonnegative real numbers and the initial conditions $x_{-k}, x_{-k+1}, \cdots, x_{0}$ are arbitrary nonnegative real numbers.

The Eq.(4.48) is trivial, moreover, $x_{n}=\frac{\gamma}{C}$ for all $n \geq 0$.
The Eq.(4.49), which is the change of variables $x_{n}=\frac{\gamma}{B} e^{y_{n}}$ reduces it to the linear difference equation

$$
\begin{equation*}
y_{n+1}+y_{n}-y_{n-k}=0, n=0,1,2, \ldots \tag{4.54}
\end{equation*}
$$

To prove this transformation, substitute $x_{n}=\frac{\gamma}{B} e^{y_{n}}$ in Eq.(4.49), we get

$$
\frac{\gamma}{B} e^{y_{n+1}}=\frac{\gamma \frac{\gamma}{B} e^{y_{n-k}}}{B \frac{\gamma}{B} e^{y_{n}}}
$$

thus

$$
e^{y_{n+1}}=\frac{e^{y_{n-k}}}{e^{y_{n}}}
$$

hence

$$
e^{y_{n+1}}=e^{y_{n-k}-y_{n}}
$$

then

$$
y_{n+1}=y_{n-k}-y_{n}
$$

hence

$$
y_{n+1}+y_{n}-y_{n-k}=0
$$

Note that when $k=1$, we have the following linear difference equation

$$
y_{n+1}+y_{n}-y_{n-1}=0, n=0,1,2, \ldots
$$

and its general solution is

$$
y_{n}=c_{1}\left(\frac{-1+\sqrt{5}}{2}\right)^{n}+c_{2}\left(\frac{-1-\sqrt{5}}{2}\right)^{n}
$$

where $c_{1}$ and $c_{2}$ are arbitrary.
Lemma 4.12. The equilibrium point of Eq.(4.54) is unstable when $k \geq 2$.
Proof. The proof is consequently from Theorem 2.4.

The Eq.(4.50) is reduced by change of variables $x_{n}=\frac{\beta}{C} y_{n}$ into the difference equation

$$
\begin{equation*}
y_{n+1}=\frac{y_{n}}{y_{n-k}} \tag{4.55}
\end{equation*}
$$

when $k=1$, every solution of $\operatorname{Eq}(4.55)$ is periodic with period 6 , and its solution is:

$$
\cdots, x_{-1}, x_{0}, \frac{x_{0}}{x_{-1}}, \frac{1}{x_{-1}}, \frac{1}{x_{0}}, \frac{x_{-1}}{x_{0}}, \cdots
$$

when $k>1$, the change of variable $x_{n}=\frac{\gamma}{B} e^{y_{n}}$ reduces Eq.(4.50) into the linear difference equation

$$
\begin{equation*}
y_{n+1}-y_{n}+y_{n-k}=0, n=0,1,2, \ldots \tag{4.56}
\end{equation*}
$$

Lemma 4.13. The equilibrium point of Eq.(4.56) is unstable when $k \geq 2$.
Proof. The proof is consequently from Theorem 2.4.

The solution of Eq.(4.51) is trivial. The Eq.(4.52) has nontrivial solution, and every solution is periodic with period $2(\mathrm{k}+1)$. Finally, every solution of Eq.(4.53) is periodic with period two.

## Chapter 5

## Computational Approach

### 5.1 Numerical Examples

In this section, we illustrate the results of previous sections and to support our theoretical discussions. We consider different numerical examples in this section. These examples represent different types of qualitative behavior of solutions to nonlinear difference equations.

In order to achieve the full benefits of computers, and to observe this numerical results clearly, we present both graphs and tables of solutions that were carried out using MATLAB. Different values of parameters are chosen, and It should be noted that $y_{-k}, y_{-k+1}, \ldots, y_{0}$ are also different initial points.

Example 5.1. Consider the third order difference equation when $k=2$ in Eq.(3.10)

$$
y_{n+1}=\frac{1+y_{n}+2 y_{n-2}}{y_{n}+2 y_{n-2}}, n=0,1,2, \ldots
$$

with initial conditions $y_{-2}=1, y_{-1}=2, y_{0}=3$

In previous chapter, we have proved in theorem 3.2 that $\bar{y}$ is asymptotically stable. look at the table 5.1 and figure 5.1. Observe that $\bar{y}=1.2638$ is asymptotically stable.

| n | $\mathrm{y}(\mathrm{n})$ | n | $\mathrm{y}(\mathrm{n})$ | n | $\mathrm{y}(\mathrm{n})$ | n | $\mathrm{y}(\mathrm{n})$ |
| ---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | 1 | 26 | 1.2638 | 51 | 1.2638 | 76 | 1.2638 |
| 2 | 2 | 27 | 1.2638 | 52 | 1.2638 | 77 | 1.2638 |
| 3 | 3 | 28 | 1.2638 | 53 | 1.2638 | 78 | 1.2638 |
| 4 | 1.2000 | 29 | 1.2638 | 54 | 1.2638 | 79 | 1.2638 |
| 5 | 1.1923 | 30 | 1.2638 | 55 | 1.2638 | 80 | 1.2638 |
| 6 | 1.1390 | 31 | 1.2638 | 56 | 1.2638 | 81 | 1.2638 |
| 7 | 1.2826 | 32 | 1.2638 | 57 | 1.2638 | 82 | 1.2638 |
| 8 | 1.2727 | 33 | 1.2638 | 58 | 1.2638 | 83 | 1.2638 |
| 9 | 1.2816 | 34 | 1.2638 | 59 | 1.2638 | 84 | 1.2638 |
| 10 | 1.2600 | 35 | 1.2638 | 60 | 1.2638 | 85 | 1.2638 |
| 11 | 1.2628 | 36 | 1.2638 | 61 | 1.2638 | 86 | 1.2638 |
| 12 | 1.2614 | 37 | 1.2638 | 62 | 1.2638 | 87 | 1.2638 |
| 13 | 1.2645 | 38 | 1.2638 | 63 | 1.2638 | 88 | 1.2638 |
| 14 | 1.2633 | 39 | 1.2638 | 64 | 1.2638 | 89 | 1.2638 |
| 15 | 1.2641 | 40 | 1.2638 | 65 | 1.2638 | 90 | 1.2638 |
| 16 | 1.2636 | 41 | 1.2638 | 66 | 1.2638 | 91 | 1.2638 |
| 17 | 1.2638 | 42 | 1.2638 | 67 | 1.2638 | 92 | 1.2638 |
| 18 | 1.2637 | 43 | 1.2638 | 68 | 1.2638 | 93 | 1.2638 |
| 19 | 1.2638 | 44 | 1.2638 | 69 | 1.2638 | 94 | 1.2638 |
| 20 | 1.2638 | 45 | 1.2638 | 70 | 1.2638 | 95 | 1.2638 |
| 21 | 1.2638 | 46 | 1.2638 | 71 | 1.2638 | 96 | 1.2638 |
| 22 | 1.2638 | 47 | 1.2638 | 72 | 1.2638 | 97 | 1.2638 |
| 23 | 1.2638 | 48 | 1.2638 | 73 | 1.2638 | 98 | 1.2638 |
| 24 | 1.2638 | 49 | 1.2638 | 74 | 1.2638 | 99 | 1.2638 |
| 25 | 1.2638 | 50 | 1.2638 | 75 | 1.2638 | 100 | 1.2638 |

Table 5.1: Solution of $\mathrm{DE} y_{n+1}=\frac{1+y_{n}+2 y_{n-2}}{y_{n}+2 y_{n-2}}$


Figure 5.1: Plot of $y_{n+1}=\frac{1+y_{n}+2 y_{n-2}}{y_{n}+2 y_{n-2}}$.

Example 5.2. Consider the fourth order difference equation when $k=3$ in Eq.(3.10)and parameters: $p=2, l=3$, and $q=4$

$$
y_{n+1}=\frac{2+y_{n}+3 y_{n-3}}{y_{n}+4 y_{n-3}}, n=0,1,2, \ldots
$$

with initial conditions $y_{-3}=1, y_{-2}=1.2, y_{-1}=1.1, y 0=1.15$

Observe that $p+l>q$ and $y_{-3}, y_{-2}, y_{-1}, y 0 \in[1,1.25]$. then by theorem $3.3 y_{n} \in$ [ $1,1.25]$ for all $n=0,1,2, \ldots$ this case $\bar{y}=1.1483$ For Fortunately, the computational result emphasis the theoretical result. Look at the table 5.2 and figure 5.2.

Example 5.3. Consider the third order difference equation when $k=2$ in Eq.(4.23)with parameters $B=2$ and $C=3$

$$
y_{n+1}=\frac{2}{y_{n}}+\frac{3}{y_{n-2}}
$$

and initial conditions $y_{-2}=1, y_{-1}=2, y_{0}=3$.
Example 5.4. In section 3.6, we investigated the existence of two cycles, and theorem 3.4 investigated the conditions of existence of two cycles. Consider the fourth order difference equation when $k=3$ in Eq. (3.3) with parameters $p=3, l=7$, and $q=0.4$

$$
y_{n+1}=\frac{3+y_{n}+7 y_{n-3}}{y_{n}+0.4 y_{n-3}}
$$

| n | $\mathrm{y}(\mathrm{n})$ | n | $\mathrm{y}(\mathrm{n})$ | n | $\mathrm{y}(\mathrm{n})$ | n | $\mathrm{y}(\mathrm{n})$ |
| ---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | 1 | 26 | 1.1484 | 51 |  | 76 |  |
| 2 | 1.2000 | 27 | 1.1483 | 52 |  | 77 |  |
| 3 | 1.1000 | 28 | 1.1483 | 53 |  | 78 |  |
| 4 | 1.1500 | 29 | 1.1484 | 54 |  | 79 |  |
| 5 | 1.1942 | 30 | 1.1483 | 55 |  | 80 |  |
| 6 | 1.1335 | 31 | 1.1483 | 56 |  | 81 |  |
| 7 | 1.1626 | 32 | 1.1483 | 57 |  | 82 |  |
| 8 | 1.1475 | 33 | 1.1483 | 58 |  | 83 |  |
| 9 | 1.1360 | 34 | 1.1483 | 59 |  | 84 |  |
| 10 | 1.1528 | 35 | 1.1483 | 60 |  | 85 |  |
| 11 | 1.1443 | 36 | 1.1483 | 61 |  | 86 |  |
| 12 | 1.1487 | 37 | 1.1483 | 62 |  | 87 |  |
| 13 | 1.1518 | 38 | 1.1483 | 63 |  | 88 |  |
| 14 | 1.1470 | 39 | 1.1483 | 64 |  | 89 |  |
| 15 | 1.1495 | 40 | 1.1483 | 65 |  | 90 |  |
| 16 | 1.1482 | 41 |  | 66 |  | 91 |  |
| 17 | 1.1474 | 42 |  | 67 |  | 92 |  |
| 18 | 1.1487 | 43 |  | 68 |  | 93 |  |
| 19 | 1.1480 | 44 |  | 69 |  | 94 |  |
| 20 | 1.1480 | 45 |  | 70 |  | 95 |  |
| 21 | 1.1486 | 46 |  | 71 |  | 96 | 1.1483 |
| 22 | 1.1482 | 47 |  | 72 |  | 97 | 1.1483 |
| 23 | 1.1484 | 48 |  | 73 |  | 98 | 1.1483 |
| 24 | 1.1483 | 49 |  | 74 |  | 99 | 1.1483 |
| 25 | 1.1483 | 50 |  | 75 |  | 100 | 1.1483 |

Table 5.2: Solution of $\mathrm{DE} y_{n+1}=\frac{2+y_{n}+3 y_{n-3}}{y_{n}+4 y_{n-3}}$


Figure 5.2: Plot of $y_{n+1}=\frac{2+y_{n}+3 y_{n-3}}{y_{n}+4 y_{n-3}}$.


Figure 5.3: Plot of $y_{n+1}=\frac{2}{y_{n}}+\frac{3}{y_{n-2}}$

| n | $\mathrm{y}(\mathrm{n})$ | n | $\mathrm{y}(\mathrm{n})$ | n | $\mathrm{y}(\mathrm{n})$ | n | $\mathrm{y}(\mathrm{n})$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | 1 | 26 | 2.2194 | 51 | 2.2539 | 76 | 2.2184 |
| 2 | 2 | 27 | 2.2516 | 52 | 2.2184 | 77 | 2.2539 |
| 3 | 4 | 28 | 2.2164 | 53 | 2.2539 | 78 | 2.2184 |
| 4 | 3.5 | 29 | 2.2541 | 54 | 2.2184 | 79 | 2.2539 |
| 5 | 2.0714 | 30 | 2.2196 | 55 | 2.2539 | 80 | 2.2184 |
| 6 | 1.7155 | 31 | 2.2546 | 56 | 2.2184 | 81 | 2.2539 |
| 7 | 2.0230 | 32 | 2.2180 | 57 | 2.2539 | 82 | 2.2184 |
| 8 | 2.4369 | 33 | 2.2533 | 58 | 2.2184 | 83 | 2.2539 |
| 9 | 2.5695 | 34 | 2.2182 | 59 | 2.2539 | 84 | 2.2184 |
| 10 | 2.2613 | 35 | 2.2542 | 60 | 2.2184 | 85 | 2.2539 |
| 11 | 2.1155 | 36 | 2.2186 | 61 | 2.2539 | 86 | 2.2184 |
| 12 | 2.1130 | 37 | 2.2539 | 62 | 2.2184 | 87 | 2.2539 |
| 13 | 2.2732 | 38 | 2.2182 | 63 | 2.2539 | 88 | 2.2184 |
| 14 | 2.2979 | 39 | 2.2538 | 64 | 2.2184 | 89 | 2.2539 |
| 15 | 2.2901 | 40 | 2.2184 | 65 | 2.2539 | 90 | 2.2184 |
| 16 | 2.1930 | 41 | 2.2540 | 66 | 2.2184 | 91 | 2.2539 |
| 17 | 2.2175 | 42 | 2.2184 | 67 | 2.2539 | 92 | 2.2184 |
| 18 | 2.2119 | 43 | 2.2539 | 68 | 2.2184 | 93 | 2.2539 |
| 19 | 2.2722 | 44 | 2.2183 | 69 | 2.2539 | 94 | 2.2184 |
| 20 | 2.2331 | 45 | 2.2539 | 70 | 2.2184 | 95 | 2.2539 |
| 21 | 2.2519 | 46 | 2.2184 | 71 | 2.2539 | 96 | 2.2184 |
| 22 | 2.2084 | 47 | 2.2539 | 72 | 2.2184 | 97 | 2.2539 |
| 23 | 2.2490 | 48 | 2.2184 | 73 | 2.2539 | 98 | 2.2184 |
| 24 | 2.2215 | 49 | 2.2539 | 74 | 2.2184 | 99 | 2.2539 |
| 25 | 2.2587 | 50 | 2.2184 | 75 | 2.2539 | 100 | 2.2184 |

Table 5.3: Solution of $y_{n+1}=\frac{2}{y_{n}}+\frac{3}{y_{n-2}}$
and initial conditions $y_{-3}=2, y_{-2}=5, y_{-1}=8$, and $y_{0}=9$. Look at the table 5.4 and figure 5.4 and observe that the solution converges to two-cycle $=\{2.3765,12.6235\}$.

| n | $\mathrm{y}(\mathrm{n})$ | n | $\mathrm{y}(\mathrm{n})$ | n | $\mathrm{y}(\mathrm{n})$ | n | $\mathrm{y}(\mathrm{n})$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | 2 | 26 | 11.0089 | 51 | 2.4340 | 276 | 12.6235 |
| 2 | 5 | 27 | 2.9170 | 52 | 12.4743 | 277 | 2.3765 |
| 3 | 8 | 28 | 11.2464 | 53 | 2.4234 | 278 | 12.6235 |
| 4 | 9 | 29 | 2.8508 | 54 | 12.5017 | 279 | 2.3765 |
| 5 | 2.6531 | 30 | 11.4294 | 55 | 2.4148 | 280 | 12.6235 |
| 6 | 8.7368 | 31 | 2.7666 | 56 | 12.5241 | 281 | 2.3765 |
| 7 | 5.6746 | 32 | 11.6297 | 57 | 2.4077 | 282 | 12.6235 |
| 8 | 7.7281 | 33 | 2.7083 | 58 | 12.5425 | 283 | 2.3765 |
| 9 | 3.3335 | 34 | 11.7737 | 59 | 2.4019 | 284 | 12.6235 |
| 10 | 9.8841 | 35 | 2.6505 | 60 | 12.5575 | 285 | 2.3765 |
| 11 | 4.3283 | 36 | 11.9219 | 61 | 2.3972 | 286 | 12.6235 |
| 12 | 8.2788 | 37 | 2.6051 | 62 | 12.5698 | 287 | 2.3765 |
| 13 | 3.6010 | 38 | 12.0336 | 63 | 2.3933 | 288 | 12.6235 |
| 14 | 10.0322 | 39 | 2.5651 | 64 | 12.5798 | 289 | 2.3765 |
| 15 | 3.6835 | 40 | 12.1379 | 65 | 2.3902 | 290 | 12.6235 |
| 16 | 9.2402 | 41 | 2.5322 | 66 | 12.5879 | 291 | 2.3765 |
| 17 | 3.5061 | 42 | 12.2206 | 67 | 2.3876 | 292 | 12.6235 |
| 18 | 10.2050 | 43 | 2.5045 | 68 | 12.5946 | 293 | 2.3765 |
| 19 | 3.3386 | 44 | 12.2926 | 69 | 2.3856 | 294 | 12.6235 |
| 20 | 10.0957 | 45 | 2.4815 | 70 | 12.6000 | 295 | 2.3765 |
| 21 | 3.2734 | 46 | 12.3512 | 71 | 2.3839 | 296 | 12.6235 |
| 22 | 10.5648 | 47 | 2.4626 | 72 | 12.6044 | 297 | 2.3765 |
| 23 | 3.1037 | 48 | 12.4005 | 73 | 2.3825 | 298 | 12.6235 |
| 24 | 10.7496 | 49 | 2.4469 | 74 | 12.6079 | 299 | 2.3765 |
| 25 | 3.0404 | 50 | 12.4409 | 75 | 2.3814 | 300 | 12.6235 |

Table 5.4: Solution of $y_{n+1}=\frac{3+y_{n}+7 y_{n-3}}{y_{n}+0.4 y_{n-3}}$

Example 5.5. Consider the third order difference equation when $k=2$ in Eq.(4.52)

$$
y_{n+1}=\frac{p}{x_{n-2}}, n=0,1,2, \ldots
$$

where $p=\frac{\alpha}{C}$


Figure 5.4: Plot of $y_{n+1}=\frac{3+y_{n}+7 y_{n-3}}{y_{n}+0.4 y_{n-3}}$

In Sec.(4.3), We proved that there is no trivial solution, and every nontrivial solution is periodic with prime period2(k+1). In this case the solution should be with period 6 . Look at the table 5.5 and Figure 5.5, we note that the solution periodic with period 6 . That is, the 6 -cycle $=\{3,2,4,1.6667,2.5,1.25\}$.


Figure 5.5: Plot of $y_{n+1}=\frac{5}{y_{n-2}}$

| N | $\mathrm{X}(\mathrm{n})$ | N | $\mathrm{X}(\mathrm{n})$ | N | $\mathrm{X}(\mathrm{n})$ | N | $\mathrm{X}(\mathrm{n})$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | 3 | 26 | 2 | 51 | 4 | 76 | 1.6667 |
| 2 | 2 | 27 | 4 | 52 | 1.6667 | 77 | 2.5 |
| 3 | 4 | 28 | 1.6667 | 53 | 2.5 | 78 | 1.25 |
| 4 | 1.6667 | 29 | 2.5 | 54 | 1.25 | 79 | 3 |
| 5 | 2.5 | 30 | 1.25 | 55 | 3 | 80 | 2 |
| 6 | 1.25 | 31 | 3 | 56 | 2 | 81 | 4 |
| 7 | 3 | 32 | 2 | 57 | 4 | 82 | 1.6667 |
| 8 | 2 | 33 | 4 | 58 | 1.6667 | 83 | 2.5 |
| 9 | 4 | 34 | 1.6667 | 59 | 2.5 | 84 | 1.25 |
| 10 | 1.6667 | 35 | 2.5 | 60 | 1.25 | 85 | 3 |
| 11 | 2.5 | 36 | 1.25 | 61 | 3 | 86 | 2 |
| 12 | 1.25 | 37 | 3 | 62 | 2 | 87 | 4 |
| 13 | 3 | 38 | 2 | 63 | 4 | 88 | 1.6667 |
| 14 | 2 | 39 | 4 | 64 | 1.6667 | 89 | 2.5 |
| 15 | 4 | 40 | 1.6667 | 65 | 2.5 | 90 | 1.25 |
| 16 | 1.6667 | 41 | 2.5 | 66 | 1.25 | 91 | 3 |
| 17 | 2.5 | 42 | 1.25 | 67 | 3 | 92 | 2 |
| 18 | 1.25 | 43 | 3 | 68 | 2 | 93 | 4 |
| 19 | 3 | 44 | 2 | 69 | 4 | 94 | 1.6667 |
| 20 | 2 | 45 | 4 | 70 | 1.6667 | 95 | 2.5 |
| 21 | 4 | 46 | 1.6667 | 71 | 2.5 | 96 | 1.25 |
| 22 | 1.6667 | 47 | 2.5 | 72 | 1.25 | 97 | 3 |
| 23 | 2.5 | 48 | 1.25 | 73 | 3 | 98 | 2 |
| 24 | 1.25 | 49 | 3 | 74 | 2 | 99 | 4 |
| 25 | 3 | 50 | 2 | 75 | 4 | 100 | 1.6667 |

Table 5.5: Solution of DE $x_{n+1}=\frac{5}{x_{n-2}}$

### 5.2 Phase Space Diagram

Phase space (also known as state space) is the set of all possible states of a dynamical system. Since it is usually impossible to derive an explicit formula for the solution of a nonlinear equation except for a few types which have been introduce in section (1.5). The phase space provides an extremely useful way for visualizing and understanding qualitative features of solutions. In this section we introduce phase state diagrams of some difference equations and compare these graphs with time series graphs for the same of difference equations. The following example we present a convergence solution.

Example 5.6. Consider the fourth order difference equation when $k=3$.

$$
y_{n+1}=\frac{2+y_{n}+4 y_{n-3}}{y_{n}+4 y_{n-3}}
$$

Figure (5.6) illustrates phase state diagram for four sets of initial values.
The next example illustrates a divergence sequence.
Example 5.7. Consider the fourth order difference equation when $k=3$.

$$
y_{n+1}=\frac{3+y_{n}+7 y_{n-3}}{y_{n}+0.4 y_{n-3}}
$$

Figure (5.7) illustrate phase state diagram while figure (5.8) illustrates time series solution.


Figure 5.6: Phase state graph of $y_{n+1}=\frac{2+y_{n}+4 y_{n-3}}{y_{n}+4 y_{n-3}}$


Figure 5.7: Phase state graph of $y_{n+1}=\frac{3+y_{n}+7 y_{n-3}}{y_{n}+0.4 y_{n-3}}$


Figure 5.8: Time series solution $y_{n+1}=\frac{3+y_{n}+7 y_{n-3}}{y_{n}+0.4 y_{n-3}}$

### 5.3 Matlab Program

The mfile function investigate the the nonlinear rational difference equation: 3.3

$$
x_{n+1}=\frac{p+x_{n}+l x_{n-k}}{x_{n}+q x_{n-l}}, n=0,1,2, \cdots
$$

where the parameters $p, q$ and initial conditions $x_{-k}, x_{-k+1}, \ldots, x_{0}$ are nonnegative real numbers, $k=\{1,2,3, \ldots\}$

There have been many good programs on Dynamical systems and Difference equations that concentrate on one or two aspects, but our program is really fantastic for these reasons :

1. Our program is user friendly.
2. The program calculate the equilibrium point.
3. You need not to write parameters when you invoke the program.
4. It give a readable out put, i.e. the out put appear in a table.
5. It produce an excellent plots that represent a solution of Difference equation.
6. It deals with an arbitrary $k$. I believe it is one of the most advantages of our program, because every time the user run the program can enter the value of $k$ which make the program more reliable.
7. Finally, this program can be easily modified for a new type of Difference equations.

For more information, see Appendix.
Remark 5.1. The mfile in A. 1 that has been reported is designed to solve the rational difference equation

$$
y_{n+1}=\frac{p+y_{n}+l y_{n-k}}{y_{n}+q y_{n-k}}, n=0,1, \ldots
$$

where the parameters $p, q$, and are nonnegative real numbers and the initial conditions $y_{-k}, y_{-k+1}, \cdots, y_{0}$ are arbitrary nonnegative real numbers.

Hence, if we want solve an equation that rises from special cases in ch.4, a slightly modification is needed. Furthermore, we need to modify the difference equation and the inserting parameters code.

## Appendix A

## Appendix

## A. 1 Rational Difference Equation Program

\% A File to the rational difference equations of order k
\% call as: ratdiff
\%You have to enter parameters: $\mathrm{p}, \mathrm{l}, \mathrm{q}$
\% and initial values: y
\%This program solves the equation :
$\% \mathrm{Xn}+1=\left(\mathrm{p}+\mathrm{Xn}+\mathrm{l}^{*} \mathrm{Xn}-\mathrm{k}\right) /\left(\mathrm{Xn}+\mathrm{q}^{*} \mathrm{Xn}-\mathrm{k}\right)$
\% numerically producing a table and graph of the solution
\% It has been designed to deal with arbitrary k
function ratdiff;
$\mathrm{k}=$ input('enter the value of the positive integer $\mathrm{k}=$ ');
$\mathrm{p}=$ input('enter the value of the positive parameter $\mathrm{p}=$ ');

$\mathrm{q}=\operatorname{input}($ 'enter the value of the positive parameter $\mathrm{q}=$ ');
solution $=$ ddifkk $(\mathrm{k}, \mathrm{p}, \mathrm{l}, \mathrm{q})$;
disp(' ')
disp(' Table ')
$\operatorname{disp}\left({ }^{\prime}\right)$
disp('The solution $\mathrm{x}(\mathrm{n})$ is given in the following table : ')
d=[solution(1:25,:),solution(26:50,:),solution(51:75,:),solution(276:300,.:)];
disp(
$\operatorname{disp}\left({ }^{\prime} \mathrm{n} \mathrm{x}(\mathrm{n}) \mathrm{n} \mathrm{x}(\mathrm{n}) \mathrm{n} \mathrm{x}(\mathrm{n}) \mathrm{n} \mathrm{x}(\mathrm{n})^{\prime}\right)$
disp(
disp(d)
fixedpoint $=\left(\left((1+\mathrm{l})+\operatorname{sqrt}\left((1+\mathrm{l}) \wedge 2+4^{*} \mathrm{p}^{*}(1+\mathrm{q})\right)\right) /\left(2^{*}(1+\mathrm{q})\right)\right)$;
fprintf('fixedpoint $=\% 2.4 \mathrm{f} . \mathrm{n}$ ',fixedpoint);
function plotandeval=ddifkk(k,p,l,q);
\% Give an initial values for $\mathrm{y}(-\mathrm{k}) \ldots \mathrm{y}(0)$
for $\mathrm{i}=1: \mathrm{k}+1$;
$\mathrm{x}(\mathrm{i})=\operatorname{input}($ 'Enter the value of the positive initial condition $\mathrm{x}=$ ');
end
for $\mathrm{n}=\mathrm{k}+1$ :300
$\mathrm{x}(\mathrm{n}+1)=\left(\mathrm{p}+\mathrm{x}(\mathrm{n})+\mathrm{l}^{*} \mathrm{x}(\mathrm{n}-\mathrm{k})\right) /\left(\mathrm{x}(\mathrm{n})+\mathrm{q}^{*} \mathrm{x}(\mathrm{n}-\mathrm{k})\right) ;$
end
$\mathrm{t}=1: 301$;
plotandeval $=[t ; \mathrm{x}]^{\prime}$;
grid on
hold on
$\mathrm{t}=1: 301$;
plot(t,x,'b.-');
xlabel('n-iteration');
ylabel('Y(n)');
title('plot of $\mathrm{y}(\mathrm{n}+1)=\left(\mathrm{p}+\mathrm{y}(\mathrm{n})+\mathrm{l}^{*} \mathrm{y}(\mathrm{n}-\mathrm{k})\right) /\left(\mathrm{y}(\mathrm{n})+\mathrm{q}^{*} \mathrm{y}(\mathrm{n}-\mathrm{l})^{\prime}\right)$;
$\mathrm{p} 1=\operatorname{strcat}\left(\mathrm{k}={ }^{\prime}\right.$, num $\left.2 \operatorname{str}(\mathrm{k})\right)$;
$\mathrm{p} 2=\operatorname{strcat}\left(\mathrm{p}={ }^{\prime}\right.$, num2str( p$),{ }^{\prime}, \mathrm{q}=$ ', num $2 \operatorname{str}(\mathrm{q}),{ }^{\prime}, \mathrm{l}=$ ' , num $\left.2 \operatorname{str}(\mathrm{l})\right)$;
legend(p1,p2);

## A. 2 Phase Space Diagram Program

function ratdiffphas;
$\mathrm{k}=$ input('enter the value of the positive integer $\mathrm{k}=$ ');
$\mathrm{p}=$ input('enter the value of the positive parameter $\mathrm{p}=$ ');
$\mathrm{l}=\mathrm{input}($ 'enter the value of the positive parameter $\mathrm{L}=$ ');
$\mathrm{q}=$ input('enter the value of the positive parameter $\mathrm{q}=$ ');
$x n=z \operatorname{eros}(1,300)$;

```
xnn=zeros(1,300);
fixedpoint=(((1+l)+\operatorname{sqrt}((1+l)^2+4* (p* (1+q)))/(2* (1+q)));
fprintf('fixedpoint =
for i=1:k+1;
x(i)=input('enter the value of the positive initial condition x =');
end for j=1:k;
y(j)=x(j+1);
end
for n=k+1:299
y(n)=(p+x(n)+l*x(n-k))/(x(n)+q*x(n-k));
x(n+1)=y(n);
end
y(300)=(p+x(300)+l*x(300-k))/(x(300)+q*x(300-k));
grid on
hold on
plot(x,y,'b.-')
xlabel('y(n)')
ylabel('y(n+1)')
title('plot of y(n+1)=(p+y(n)+\mp@subsup{l}{}{*}y(n-k))/(y(n)+\mp@subsup{q}{}{*}y(n-l)}\mp@subsup{)}{}{\prime})
p1=strcat('k= ',num2str(k));
```

$\mathrm{p} 2=\operatorname{strcat}\left(\mathrm{p}={ }^{\prime}\right.$, ,num $2 \operatorname{str}(\mathrm{p}),{ }^{\prime}, \mathrm{q}={ }^{\prime}$, num $2 \operatorname{str}(\mathrm{q}),{ }^{\prime}, \mathrm{l}={ }^{\prime}$, num $\left.2 \operatorname{str}(\mathrm{l})\right) ;$ legend(p1,p2);

## A. 3 Cobweb Diagram Program

Cobweb Diagram
First Order Difference Equations A proc to generate a sequence of iterates of the difference equation $>$ restart;with(plots):setoptions(thickness=2):

Define a function (procedure) that will generate the iterates of a function $g$ local i;
$\operatorname{seq}((\mathrm{g} @ i)(p 0), i=0 . . n \max )$
end:
Define the data and the function h where $\mathrm{x}(\mathrm{n}+1)=\mathrm{h}(\mathrm{x}(\mathrm{n}))$
$>\mathrm{r}:=3.55:$
$\mathrm{h}:=\mathrm{x}->\mathrm{r}^{*} \mathrm{x}^{*}(1-\mathrm{x})$;
$>$ data: $=[$ iterates $(\mathrm{h}, 0.1,30)]$ :
$>$ datapoint : $=[\operatorname{seq}([\mathrm{n}-1, \operatorname{data}[\mathrm{n}]], \mathrm{n}=1 . .30)]$ :
Plot the time dependent behavior
$>\mathrm{rr}:=\operatorname{convert}(\mathrm{r}$, string $)$ :
code $:=\operatorname{cat}\left({ }^{‘}\right.$ Discrete Logistic - $\mathrm{r}={ }^{\text {' }}$,rr);
$\operatorname{plot}($ datapoint $, \mathrm{x}=0 . .30,0 . .1$, title $=$ code );

A proc to generate a cobweb graph of an iteration - lastn even
$>$ cobweb:= proc(g,t1,lastn)
local pp, pp1, i, plot1, plot2;
$\mathrm{pp} 1:=[\operatorname{seq}((\mathrm{g} @ @(\operatorname{trunc}((\mathrm{i}+2) / 4)))(\mathrm{t} 1), \mathrm{i}=1 .$. lastn $)] ;$
$\mathrm{pp}:=\left[\operatorname{seq}\left(\left[\operatorname{pp1}\left[2^{*} \mathrm{i}-1\right], \operatorname{pp} 1\left[2^{*} \mathrm{i}\right]\right], \mathrm{i}=1 . . \operatorname{lastn} / 2\right)\right] ;$
$\operatorname{plot} 1:=\operatorname{plot}(\mathrm{pp}, \mathrm{x}=0 . .1):$
plot2: $=\operatorname{plot}(\mathrm{x}, \mathrm{g}(\mathrm{x}), \mathrm{x}=0 . .1$, color=black $)$ :
plots[display](plot1,plot2);
end:
Execute the cobweb procedure
$>$ cobweb(h, $0.2,30)$;
\%End the program

## Bibliography

[1] S. Abu-Baha', Dynamics of a $k^{t h}$ order Rational Difference Equation Using Theoretical and Computational Approach, Master thesis, Birzeit University, 2005.
[2] R. Abu-Saris, R. DeVault, Global stability of $y_{n+1}=A+\frac{y_{n}}{y_{n-k}}$, Appl. Math. 16 (2003) 173-178.
[3] Mehdi Dehghan, Majid Jaberi Douraki, Marjan Jaberi Douraki, Dynamics of a rational difference equations using both theoretical and computational approaches, Applied Mathematics and Computation, Article in press, 2004.
[4] Mahdi Dehghan, Reza Mazrooei-Sebdani, Dynamics of a higher rational difference equation, Appl. Math. Comp. 178 (2006) 345-354.
[5] R. DeVault, W. Kosmala,G. Ladas, S.W. Schuults, Global behavior of $y_{n+1}=$ $\frac{p+y_{n-k}}{q y_{n}+y_{n-k}}$, Nonlinear Analysis 47, 4743-4751, 2001
[6] M.M. El-Afifi, On the recursive sequence $x_{n+1}=\frac{\alpha+\beta x_{n}+\gamma x_{n-1}}{B x_{n}+C x_{n-1}}$, Applied Mathematics and Computations,147 (2004) 617-628.
[7] S. Elaydi, Introduction to difference equations, Springer-NewYork, 1996.
[8] S. Elaydi, Discrete Chaos, Springer-Newyork, 2000.
[9] H. El-Owaidy, A. Ahmed, M. Mousa, On asymptotic behavior of the difference equatin $x_{n+1}=\alpha+\frac{x_{n-k}}{x_{n}}$, Appl. Math. Comp. 147 (2004) 163-167.
[10] M. Jaberi Douraki, M. Dehghan, M. Razzaghi, The qualitative behavior of solutions of a non-linear difference equation, Appl. Math., Comp. 170(2005) 485-502.
[11] M.R.S Kulenovic, G. Ladas, Dynamics of the second Order Rational Difference Equations: with Open Problems and Conjectures, Chapman \& Hall/CRC, Boca Raton, 2001.
[12] M.R.S Kulenovic, G. Ladas, L.F. Martins, and I.W. Rudrigues, The Dynamics of $x_{n+1}=\frac{\alpha+\beta x_{n}}{A+C x_{n-1}}$, Facts and Conjectures,An international journal:Computers and Mathematics with applications,2003.
[13] M.Saleh, M. Aloqeili, On the rational difference equation $y_{n+1}=A+\frac{y_{n-k}}{y_{n}}$, Appl. Math. Comp. 171 (2005) 862-869.
[14] M.Saleh, M. Aloqeili, On the rational difference equation $y_{n+1}=A+\frac{y_{n-k}}{y_{n}}$ with $A<0$, Appl. Math. Comp. 176 (2006) 359-363.
[15] M.Saleh, M. Aloqeili, On the rational difference equation $y_{n+1}=A+\frac{y_{n}}{y_{n-k}}$, Appl. Math. Comp. 177 (2006) 189-193.
[16] Wan-Tong Li, Hong-Rui Sun, Dynamics of a Rational difference equation, Appl. Math. 163 (2005) 577-591.

